1.5 Concept Questions

1. a. $m_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{f(2+h) - f(2)}{h}$

b. $m_{tan} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$

2. a. $r_{av} = \frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{f(2+h) - f(2)}{h}$

b. $r_{inst} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$

c. They are the same.

10. a. $m_{sec} = \frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{\left[(2+h)^2 - (2+h)\right] - \left(2^2 - 2\right)}{h} = \frac{h^2 + 3h}{h} = 3 + h$

b. $m_{tan} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{(2+h) - 2} = \lim_{h \to 0} (3 + h) = 3$

c. $y - 2 = 3(x - 2) \Rightarrow y = 3x - 4$

14. a. $m_{sec} = \frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{\frac{1}{(1+h)+1} - \frac{1}{1+1}}{(1+h) - 1} = \frac{2 - (2+h)}{2h(2+h)} = \frac{-h}{2h(2+h)} = -\frac{1}{2(2+h)}$

b. $m_{tan} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{(1+h) - 1} = \lim_{h \to 0} \left[-\frac{1}{2(2+h)}\right] = -\frac{1}{4}$

c. $y - \frac{1}{2} = -\frac{1}{4}(x - 1) \Rightarrow y = -\frac{1}{4}x + \frac{3}{4}$

16. $\lim_{h \to 0} \frac{g(-1+h) - g(-1)}{(-1+h) - (-1)} = \lim_{h \to 0} \frac{\left[(-1+h)^2 - (-1+h) + 2\right] - \left[(-1)^2 - (-1) + 2\right]}{h}$

$= \lim_{h \to 0} \frac{h^2 - 2h + 1 - h + 1 + 2 - 4}{h} = \lim_{h \to 0} \frac{h(h-3)}{h} = -3$

18. $\lim_{h \to 0} \frac{f(4+h) - f(4)}{(4+h) - 4} = \lim_{h \to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h} = \lim_{h \to 0} \frac{\left(\sqrt{4+h} - 2\right)\left(\sqrt{4+h} + 2\right)}{h\left(\sqrt{4+h} + 2\right)} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{4+h} + 2\right)} = \frac{1}{4}$

20. $\lim_{h \to 0} \frac{f(1+h) - f(1)}{(1+h) - 1} = \lim_{h \to 0} \frac{\frac{1}{(1+h) - 2} - \frac{1}{1-2}}{h} = \lim_{h \to 0} \frac{\frac{1}{h-1} + 1}{h} = \lim_{h \to 0} \frac{1 + h - 1}{h(h-1)} = -1$

36. Using the definition of the derivative, we find $f(x) = 2x^{1/4}$ and $a = 16$.

38. Using the definition of the derivative, we find $f(x) = 2^x$ and $a = 3$.

40. Using the definition of the derivative with $h = x - \frac{\pi}{2}$, we find $f(x) = \sin x$ and $a = \frac{\pi}{2}$.

41. True. The slope of the secant line passing through $(a, f(a))$ and $(b, f(b))$ is $m = \frac{f(b) - f(a)}{b - a}$. By definition, the average rate of change of $f(x)$ over $[a, b]$ is $r_{av} = \frac{f(b) - f(a)}{b - a}$.

42. True. Consider the function $f(x) = mx + b$ whose graph is a straight line. Since the tangent line to the graph of f at any point is the line itself, the tangent line intersects the graph of f at infinitely many points.

43. False. If the tangent line exists at a point $(x_0, f(x_0))$, then $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$ exists and must be unique.

44. True. The slope of the tangent line to the graph of $f(x)$ at $x = a$ is given by $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$. Put $x = a + h$. Then $h = x - a$, and since $x \to a$ as $h \to 0$, $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

Chapter 1 Review

Concept Review

1. a. L, f, L, a **b.** right **c.** exist, L **d.** $\varepsilon > 0$, $\delta > 0$

2. a. $\lim_{x \to a} [f(x) \pm g(x)] = L \pm M$, $\lim_{x \to a} [f(x) g(x)] = LM$, $\lim_{x \to a} [cf(x)] = cL$, $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$

b. $p(a)$

c. $r(x)$

3. $\lim_{x \to a} g(x) = L$

4. a. continuous **b.** removable **c.** jump **d.** left

5. a. $(-\infty, \infty)$ **b.** its domain **c.** continuous

6. a. $[a, b]$, $f(c) = M$ **b.** $f(x) = 0$, (a, b)

7. a. $m_{\tan} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$ **b.** $y - f(a) = m_{\tan}(x - a)$

8. a. $\frac{f(a + h) - f(a)}{h}$ **b.** $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$

2.1 Concept Questions

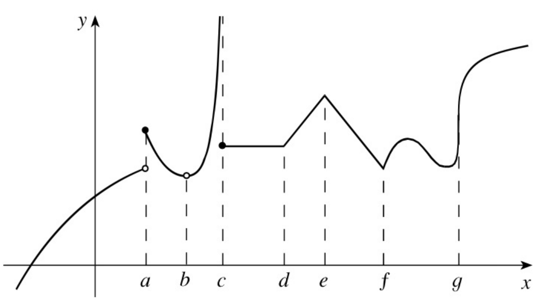
1. a. i. It gives the slope of the secant line passing through the points $(x, f(x))$ and $(x + h, f(x + h))$.

ii. It gives the average rate of change of f over the interval $[x, x + h]$.

b. i. It gives the slope of the tangent line to the graph of f at the point $(x, f(x))$.

ii. It gives the instantaneous rate of change of f at x.

2. Loosely speaking, a function f does not have a derivative at a if the graph of f does not have a tangent line at a, or if it has a vertical tangent line at a. The function whose graph is shown in the figure fails to be differentiable at $x = a, b$, and c because it is discontinuous at each of these numbers. The derivative of the function does not exist at $x = d$, e and f because it has a kink at each point on the graph corresponding to these numbers. Finally, the function is not differentiable at $x = g$ because the tangent line is vertical at $(g, f(g))$.



4. $f(x) = 2x^2 + x \quad \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left[2(x+h)^2 + (x+h)\right] - \left(2x^2 + x\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(2x^2 + 4xh + 2h^2 + x + h\right) - \left(2x^2 + x\right)}{h} = \lim_{h\to 0} \frac{4xh + 2h^2 + h}{h} = \lim_{h\to 0} \frac{h(4x + 2h + 1)}{h}$$

$$= \lim_{h\to 0} (4x + 2h + 1) = 4x + 1 \text{ with domain } (-\infty, \infty).$$

8. $f(x) = 2\sqrt{x} \quad \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{2\sqrt{x+h} - 2\sqrt{x}}{h} = 2\lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= 2\lim_{h\to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = 2\lim_{h\to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = 2\lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{2}{2\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}} \text{ with domain } (0, \infty).$$

10. $f(x) = \dfrac{1}{x} \quad \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = \lim_{h\to 0} \frac{\dfrac{x - (x+h)}{x(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{-h}{hx(x+h)} = -\lim_{h\to 0} \frac{1}{x(x+h)} = -\frac{1}{x^2} \text{ with domain } (-\infty, 0) \cup (0, \infty).$$

12. $f(x) = -\dfrac{2}{\sqrt{x}} \quad \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{-\dfrac{2}{\sqrt{x+h}} - \left(-\dfrac{2}{\sqrt{x}}\right)}{h} = 2\lim_{h\to 0} \frac{\dfrac{-\sqrt{x} + \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}}{h}$$

$$= 2\lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h\sqrt{x}\sqrt{x+h}} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = 2\lim_{h\to 0} \frac{(x+h) - x}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= 2\lim_{h\to 0} \frac{h}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x+h} + \sqrt{x}\right)} = 2\lim_{h\to 0} \frac{1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{2}{\sqrt{x}\sqrt{x}\left(2\sqrt{x}\right)}$$

$$= \frac{1}{x\sqrt{x}} \text{ with domain } (0, \infty).$$

16. $f(x) = 3x^2 - 4x + 2 \quad \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left[3(x+h)^2 - 4(x+h) + 2\right] - \left(3x^2 - 4x + 2\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(3x^2 + 6xh + 3h^2 - 4x - 4h + 2\right) - \left(3x^2 - 4x + 2\right)}{h} = \lim_{h\to 0} \frac{h(6x + 3h - 4)}{h}$$

$$= \lim_{h\to 0} (6x + 3h - 4) = 6x - 4$$

The slope of the tangent line at $(2, 6)$ is $f'(2) = 6(2) - 4 = 8$. An equation of the tangent line is $y - 6 = 8(x - 2)$ or $y = 8x - 10$.

20. $f(x) = \dfrac{2}{x} \quad \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{2}{x+h} - \dfrac{2}{x}}{h} = 2\lim_{h\to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = 2\lim_{h\to 0} \frac{\dfrac{x - (x+h)}{x(x+h)}}{h}$$

$$= 2\lim_{h\to 0} \frac{-h}{hx(x+h)} = -2\lim_{h\to 0} \frac{1}{x(x+h)} = -\frac{2}{x^2}$$

The slope of the tangent line at $(2, 1)$ is $f'(2) = -\frac{2}{4} = -\frac{1}{2}$. An equation of the tangent line is $y - 1 = -\frac{1}{2}(x - 2)$ or $y = -\frac{1}{2}x + 2$.

44. f is not differentiable at 1 because f is not continuous at 1.

46. f is not differentiable at 2 because the graph of f has a kink at the point $(2, 0)$.

48. f is not differentiable at -2 because f is not continuous there. f also fails to be differentiable at 0 and 1, because the graph of f has kinks at the points $(0, 3)$ and $(1, 4)$.

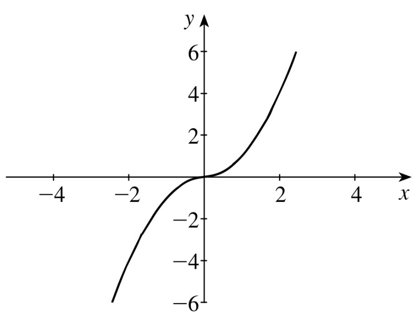
50. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+1) = 1$, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(x^2+1\right) = 1$. Therefore, $\lim_{x\to 0} f(x) = 1$. Also, $f(0) = 0 + 1 = 1$, and so $\lim_{x\to 0} f(x) = f(0)$. Therefore, f is continuous at 0.

To show that f is not differentiable at 0, let $h < 0$ and consider $\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{(h+1)-1}{h} = \lim_{h\to 0^-} 1 = 1$. Next, if $h > 0$, then $\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{\left[(0+h)^2+1\right]-1}{h} = \lim_{h\to 0^+} h = 0$. This shows that $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$ does not exist, and so by definition, f is not differentiable at 0.

52. $f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ Because $|\sin(1/x)| \le 1$, we have $0 \le |x\sin(1/x)| \le |x|$. Since $\lim_{x\to 0} |x| = 0$, the Squeeze Theorem implies that $\lim_{x\to 0} x\sin(1/x) = 0 = f(0)$, and so f is continuous at 0.

To show that f is not differentiable at 0, we compute $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h\sin(1/h)}{h} = \lim_{h\to 0} \sin(1/h)$ which does not exist. Therefore, f is not differentiable at 0.

58. a.



$f(x) = x|x| = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \ge 0 \end{cases}$

b. If $x \ge 0$, then $f(x) = x^2$, so

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h\to 0} \frac{\left(x^2+2xh+h^2\right)-x^2}{h} = \lim_{h\to 0} (2x+h) = 2x$$

If $x < 0$, then $f(x) = -x^2$, and a similar calculation shows that $f'(x) = -2x$. So f is differentiable everywhere.

c. From the results of part b, we see that $f'(x) = \begin{cases} -2x & \text{if } x < 0 \\ 2x & \text{if } x \ge 0 \end{cases}$

61. a. $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \Rightarrow$

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^2\sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} h\sin\frac{1}{h}.$$

Now $0 \le \left|h\sin\frac{1}{h}\right| \le |h|$, and the Squeeze Theorem implies that $\lim_{h\to 0} h\sin\frac{1}{h} = 0$. Therefore f is differentiable at 0 and $f'(0) = 0$.

65. True. By definition, the slope of the tangent line to the graph of f at $(3, f(3))$ is given by $f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$.

66. False. The function $f(x) = 0$ is differentiable, and the function $g(x) = |x|$ is not differentiable at 0. But the function $h = fg$ defined by $h(x) = f(x)g(x) = 0$ is differentiable everywhere.

67. False. The function $f(x) = |x|$ is not differentiable at 0 (see Example 6). Taking $f(x) = g(x) = |x|$, we see that f and g are not differentiable at 0, but the product fg defined by $(fg)(x) = f(x)g(x) = |x||x| = x^2$ is differentiable at 0.

68. False. The functions $f(x) = |x|$ and $g(x) = -|x|$ are not differentiable at 0, but $f + g$ defined by $(f + g)(x) = f(x) + g(x) = |x| - |x| = 0$ is differentiable at 0.

69. False. Consider $f(x) = \sqrt{x}$. The domain of f is $[0, \infty)$, but the domain of f' is $(0, \infty)$. (See Example 1.)

70. True. One example is the function $f(x) = |x - 1| + |x - 2| + \cdots + |x - n|$.