

7.  $f(x, y) = xy \Rightarrow \nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$  and  $g(x, y) = x^2 + 4y^2 - 1 \Rightarrow \nabla g(x, y) = 2x\mathbf{i} + 8y\mathbf{j}$ , so  $\nabla f = \lambda \nabla g \Rightarrow$

$$\left. \begin{aligned} y &= 2\lambda x & (1) \\ y\mathbf{i} + x\mathbf{j} &= 2\lambda x\mathbf{i} + 8\lambda y\mathbf{j}, \text{ and we solve } & x = 8\lambda y & (2) \\ x^2 + 4y^2 &= 1 & (3) \end{aligned} \right\} \text{Substituting (1) into (2) gives } x = 8\lambda(2\lambda x) \Rightarrow$$

$\lambda = \pm \frac{1}{4}$ . [Note that  $(x, y) \neq (0, 0)$  since this violates (3).] If  $\lambda = -\frac{1}{4}$ , then (1) gives  $y = -\frac{1}{2}x$  and so (3) gives  $y = \pm \frac{\sqrt{2}}{4}$  and  $x = \mp \frac{\sqrt{2}}{2}$ ; if  $\lambda = \frac{1}{4}$ , then (1) gives  $y = \frac{1}{2}x$  and so (3) gives  $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4})$  and  $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4})$ .

$(x, y)$	$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4})$	$(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4})$	$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4})$	$(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4})$
$f(x, y)$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

From the table, we see that  $f$  has a minimum value of  $-\frac{1}{4}$  and a maximum value of  $\frac{1}{4}$ .

9.  $f(x, y) = x^2 + xy + y^2 \Rightarrow \nabla f(x, y) = (2x + y)\mathbf{i} + (x + 2y)\mathbf{j}$  and  $g(x, y) = x^2 + y^2 - 8 \Rightarrow \nabla g(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$ ,

$$\text{so } \nabla f = \lambda \nabla g \Rightarrow (2x + y)\mathbf{i} + (x + 2y)\mathbf{j} = 2\lambda x\mathbf{i} + 2\lambda y\mathbf{j}, \text{ and we solve } \left. \begin{aligned} 2x + y &= 2\lambda x \\ x + 2y &= 2\lambda y \\ x^2 + y^2 &= 8 \end{aligned} \right\} \Rightarrow \left. \begin{aligned} x &= 2y(\lambda - 1) \\ y &= 2x(\lambda - 1) \end{aligned} \right\} \Rightarrow$$

$y = \pm x$  (because  $\lambda = 1 \Rightarrow x = y = 0$ , violating the third equation). Substituting  $y = \pm x$  into the third equation gives  $x = \pm 2$  and  $y = \pm 2$ .

$(x, y)$	$(-2, -2)$	$(-2, 2)$	$(2, -2)$	$(2, 2)$
$f(x, y)$	12	4	4	12

From the table, we see that  $f$  has a minimum value of 4 and a maximum value of 12.

13.  $f(x, y, z) = x + 2y - 2z \Rightarrow \nabla f(x, y, z) = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$  and  $g(x, y, z) = x^2 + 2y^2 + 4z^2 - 1 \Rightarrow$   
 $\nabla g(x, y, z) = 2x\mathbf{i} + 4y\mathbf{j} + 8z\mathbf{k}$ , so  $\nabla f = \lambda \nabla g \Rightarrow \mathbf{i} + 2\mathbf{j} - 2\mathbf{k} = 2\lambda x\mathbf{i} + 4\lambda y\mathbf{j} + 8\lambda z\mathbf{k}$ , and we solve

$$\left. \begin{aligned} 1 &= 2\lambda x \\ 2 &= 4\lambda y \\ -2 &= 8\lambda z \\ x^2 + 2y^2 + 4z^2 &= 1 \end{aligned} \right\} \text{We see that } x = \frac{1}{2\lambda}, y = \frac{1}{2\lambda}, \text{ and } z = -\frac{1}{4\lambda}, \text{ which we substitute into the fourth equation:}$$

$(\frac{1}{2\lambda})^2 + 2(\frac{1}{2\lambda})^2 + 4(-\frac{1}{4\lambda})^2 = 1 \Rightarrow \lambda = \pm 1$ . If  $\lambda = -1$ , then  $x = -\frac{1}{2}, y = -\frac{1}{2}$ , and  $z = \frac{1}{4}$ ; if  $\lambda = 1$ , then  $x = \frac{1}{2}, y = \frac{1}{2}$ , and  $z = -\frac{1}{4}$ . We see that the minimum value of  $f$  is  $f(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{4}) = -2$  and the maximum value of  $f$  is  $f(\frac{1}{2}, \frac{1}{2}, -\frac{1}{4}) = 2$ .

15.  $f(x, y, z) = xyz \Rightarrow \nabla f(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$  and  $g(x, y, z) = x^2 + 2y^2 + \frac{1}{2}z^2 = 6 \Rightarrow$   
 $\nabla g(x, y, z) = 2x\mathbf{i} + 4y\mathbf{j} + z\mathbf{k}$ , so  $\nabla f = \lambda \nabla g \Rightarrow yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = 2\lambda x\mathbf{i} + 4\lambda y\mathbf{j} + \lambda z\mathbf{k}$ , and we solve

$$\left. \begin{aligned} yz &= 2\lambda x \\ xz &= 4\lambda y \\ xy &= \lambda z \\ x^2 + 2y^2 + \frac{1}{2}z^2 &= 6 \end{aligned} \right\} \text{Solving the first three equations for } \lambda \text{ gives } \lambda = \frac{yz}{2x} = \frac{xz}{4y} = \frac{xy}{z}, \text{ so } x^2 = 2y^2 \text{ and}$$

$z^2 = 4y^2$ . Substituting these into the last equation in the system gives  $2y^2 + 2y^2 + \frac{1}{2}(4y^2) = 6 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$ .

Thus,  $x = \pm\sqrt{2}$  and  $z = \pm 2$ . Evaluating  $f(x, y, z)$  at  $(\pm\sqrt{2}, -1, -2)$ ,  $(\pm\sqrt{2}, -1, 2)$ ,  $(\pm\sqrt{2}, 1, -2)$ , and  $(\pm\sqrt{2}, 1, 2)$ , we see that  $f$  has a minimum value of  $-2\sqrt{2}$  and a maximum value of  $2\sqrt{2}$ .

17.  $f(x, y, z) = 2x + y \Rightarrow \nabla f(x, y, z) = 2\mathbf{i} + \mathbf{j}$ ,  $g(x, y, z) = x + y + z - 1 \Rightarrow \nabla g(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ , and  $h(x, y, z) = y^2 + z^2 - 9 \Rightarrow \nabla h(x, y, z) = 2y\mathbf{j} + 2z\mathbf{k}$ , so  $\nabla f = \lambda\nabla g + \mu\nabla h$  along with the constraints  $g(x, y, z) = 0$  and  $h(x, y, z) = 0$  give the system

$$\left. \begin{aligned} 2 &= \lambda & (1) \\ 1 &= \lambda + 2\mu y & (2) \\ 0 &= \lambda + 2\mu z & (3) \\ x + y + z &= 1 & (4) \\ y^2 + z^2 &= 9 & (5) \end{aligned} \right\} \text{From (1), (2), and (3), we obtain } \mu = -\frac{1}{2y} = -\frac{1}{z}$$

$\Rightarrow z = 2y$ . Substituting this into (5) gives  $y^2 + (2y)^2 = 5y^2 = 9 \Leftrightarrow y = \pm\frac{3\sqrt{5}}{5} \Rightarrow z = \pm\frac{6\sqrt{5}}{5}$ . Then (4) gives  $x = 1 - \left(\pm\frac{3\sqrt{5}}{5}\right) - \left(\pm\frac{6\sqrt{5}}{5}\right) = \frac{5 \mp 9\sqrt{5}}{5}$ , so  $f$  has minimum value  $f\left(\frac{5-9\sqrt{5}}{5}, \frac{3\sqrt{5}}{5}, \frac{6\sqrt{5}}{5}\right) = 2 - 3\sqrt{5}$  and maximum value  $f\left(\frac{5+9\sqrt{5}}{5}, -\frac{3\sqrt{5}}{5}, -\frac{6\sqrt{5}}{5}\right) = 2 + 3\sqrt{5}$ .

21.  $f_x(x, y) = \frac{\partial}{\partial x}(3x^2 + 2y^2 - 2x - 1) = 6x - 2 = 0$   
 $f_y(x, y) = \frac{\partial}{\partial y}(3x^2 + 2y^2 - 2x - 1) = 4y = 0$  }  $\Rightarrow x = \frac{1}{3}$  and  $y = 0$ , so  $f$  has the critical point  $\left(\frac{1}{3}, 0\right)$  in the disk

$D = \{(x, y) \mid x^2 + y^2 \leq 9\}$ . Next, we use the method of Lagrange to find the critical points of  $f$  on the boundary of  $D$ .

We write  $g(x, y) = x^2 + y^2 - 9 = 0$ . Then  $\nabla f(x, y) = (6x - 2)\mathbf{i} + 4y\mathbf{j}$  and  $\nabla g(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$ . The equation

$\nabla f = \lambda\nabla g$  and the constraint equation  $g(x, y) = 0$  give the system

$$\left. \begin{aligned} 6x - 2 &= 2\lambda x & (1) \\ 4y &= 2\lambda y & (2) \\ x^2 + y^2 &= 9 & (3) \end{aligned} \right\} \text{Equation (2) gives}$$

$y = 0$  or  $\lambda = 2$ . If  $y = 0$ , then (3) gives  $x = \pm 3$ ; if  $\lambda = 2$ , then (1) gives  $x = 1$ . Substituting this value of  $x$  into (3) gives  $y = \pm 2\sqrt{2}$ .

$(x, y)$	$\left(\frac{1}{3}, 0\right)$	$(-3, 0)$	$(3, 0)$	$(1, -2\sqrt{2})$	$(1, 2\sqrt{2})$
$f(x, y)$	$-\frac{4}{3}$	32	20	16	16

From the table, we see that  $f$  has a minimum value of  $f\left(\frac{1}{3}, 0\right) = -\frac{4}{3}$  and a maximum value of  $f(-3, 0) = 32$ .

23. The distance from the origin to a point  $(x, y, z)$  on the plane  $x + 2y + z = 4$  is  $D = \sqrt{x^2 + y^2 + z^2}$ . To minimize  $D$ , we just need to minimize  $f(x, y, z) = D^2 = x^2 + y^2 + z^2$  subject to  $g(x, y, z) = x + 2y + z - 4 = 0$ .

$\nabla f(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$  and  $\nabla g(x, y, z) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ , so  $\nabla f = \lambda\nabla g$  and  $g(x, y, z) = 0$  give the system

$$\left. \begin{aligned} 2x &= \lambda \\ 2y &= 2\lambda \\ 2z &= \lambda \\ x + 2y + z &= 4 \end{aligned} \right\} \Leftrightarrow \left. \begin{aligned} x &= \frac{1}{2}\lambda \\ y &= \lambda \\ z &= \frac{1}{2}\lambda \\ x + 2y + z &= 4 \end{aligned} \right\} \Rightarrow \frac{1}{2}\lambda + 2\lambda + \frac{1}{2}\lambda = 4, \text{ giving } \lambda = \frac{4}{3}. \text{ Therefore, the required point is } \left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right).$$

## 14.2 Iterated Integrals

## ET 13.2

3.  $\int_0^2 \int_1^4 y\sqrt{x} dy dx = \int_0^2 \left[ \int_1^4 yx^{1/2} dy \right] dx = \int_0^2 \left[ \frac{1}{2}y^2x^{1/2} \right]_{y=1}^{y=4} dx = \int_0^2 \frac{15}{2}x^{1/2} dx = 5x^{3/2} \Big|_0^2 = 10\sqrt{2}$

$$6. \int_0^{\pi/2} \int_0^{\ln 2} e^{-x} \sin y \, dx \, dy = \int_0^{\pi/2} \left[ \int_0^{\ln 2} e^{-x} \sin y \, dx \right] dy = \int_0^{\pi/2} [-e^{-x} \sin y]_{x=0}^{x=\ln 2} dy = \int_0^{\pi/2} \frac{1}{2} \sin y \, dy$$

$$= -\frac{1}{2} \cos y \Big|_0^{\pi/2} = \frac{1}{2}$$

$$11. \int_{-1}^1 \int_x^{2x} e^{x+y} \, dy \, dx = \int_{-1}^1 \left[ \int_x^{2x} e^{x+y} \, dy \right] dx = \int_{-1}^1 [e^{x+y}]_{y=x}^{y=2x} dx = \int_{-1}^1 (e^{3x} - e^{2x}) dx = \left( \frac{1}{3} e^{3x} - \frac{1}{2} e^{2x} \right) \Big|_{-1}^1$$

$$= \left( \frac{1}{3} e^3 - \frac{1}{2} e^2 \right) - \left( \frac{1}{3} e^{-3} - \frac{1}{2} e^{-2} \right) = \frac{2e^6 - 3e^5 + 3e - 2}{6e^3}$$

$$12. \int_0^{\pi} \int_{e^{-2x}}^{e^{\cos x}} \frac{\ln y}{y} \, dy \, dx = \int_0^{\pi} \left[ \int_{e^{-2x}}^{e^{\cos x}} \frac{\ln y}{y} \, dy \right] dx = \int_0^{\pi} \left[ \frac{1}{2} (\ln y)^2 \right]_{y=e^{-2x}}^{y=e^{\cos x}} dx = \frac{1}{2} \int_0^{\pi} \left[ (\ln e^{\cos x})^2 - (\ln e^{-2x})^2 \right] dx$$

$$= \frac{1}{2} \int_0^{\pi} (\cos^2 x - 4x^2) dx = \frac{1}{2} \int_0^{\pi} \left[ \frac{1}{2} + \frac{1}{2} \cos 2x - 4x^2 \right] dx$$

$$= \frac{1}{2} \left( \frac{1}{2} x + \frac{1}{4} \sin 2x - \frac{4}{3} x^3 \right) \Big|_0^{\pi} = \frac{1}{12} \pi (3 - 8\pi^2)$$

$$15. \iint_R (x \cos y + y \sin x) \, dA = \int_0^{\pi/4} \int_0^{\pi/2} (x \cos y + y \sin x) \, dx \, dy = \int_0^{\pi/4} \left[ \frac{1}{2} x^2 \cos y - y \cos x \right]_{x=0}^{x=\pi/2} dy$$

$$= \int_0^{\pi/4} \left( \frac{1}{8} \pi^2 \cos y + y \right) dy = \left( \frac{1}{8} \pi^2 \sin y + \frac{1}{2} y^2 \right) \Big|_0^{\pi/4} = \frac{1}{32} \pi^2 (2\sqrt{2} + 1)$$

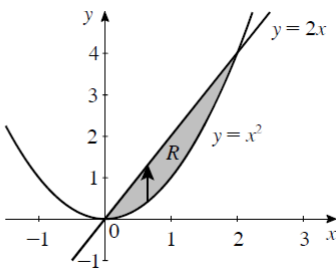
$$16. \iint_R y e^{xy} \, dA = \int_0^1 \int_0^1 y e^{xy} \, dx \, dy = \int_0^1 [e^{xy}]_{x=0}^{x=1} dy = \int_0^1 (e^y - 1) dy = (e^y - y) \Big|_0^1 = e - 2$$

$$19. \iint_R (x^3 + 2y) \, dA = \int_0^2 \int_{x^2}^{2x} (x^3 + 2y) \, dy \, dx$$

$$= \int_0^2 [x^3 y + y^2]_{y=x^2}^{y=2x} dx$$

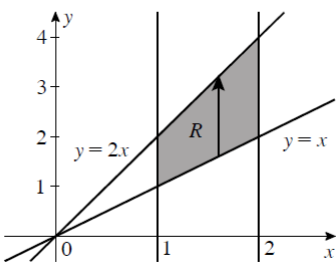
$$= \int_0^2 (x^4 + 4x^2 - x^5) dx$$

$$= \left( \frac{1}{5} x^5 + \frac{4}{3} x^3 - \frac{1}{6} x^6 \right) \Big|_0^2 = \frac{32}{5}$$

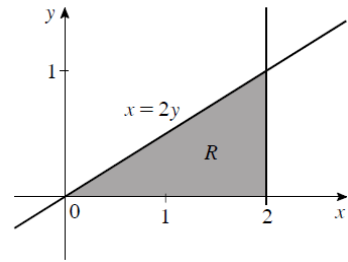


$$25. \iint_R x^2 y \, dA = \int_1^2 \int_x^{2x} x^2 y \, dy \, dx = \int_1^2 \left[ \frac{1}{2} x^2 y^2 \right]_{y=x}^{y=2x} dx$$

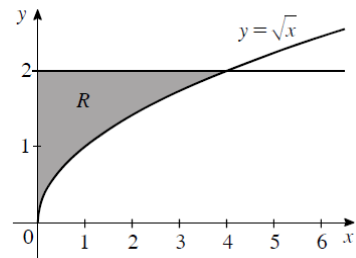
$$= \frac{1}{2} \int_1^2 3x^4 \, dx = \frac{3}{10} x^5 \Big|_1^2 = \frac{93}{10}$$



$$\begin{aligned}
 55. \int_0^1 \int_{2y}^2 e^{-x^2} dx dy &= \int_0^2 \int_0^{x/2} e^{-x^2} dy dx = \int_0^2 \left[ ye^{-x^2} \right]_{y=0}^{y=x/2} dx \\
 &= \frac{1}{2} \int_0^2 xe^{-x^2} dx = -\frac{1}{4} e^{-x^2} \Big|_0^2 = \frac{1}{4} (1 - e^{-4}) \\
 &= \frac{e^4 - 1}{4e^4}
 \end{aligned}$$



$$\begin{aligned}
 57. \int_0^4 \int_{\sqrt{x}}^2 \sin y^3 dy dx &= \int_0^2 \int_0^{y^2} \sin y^3 dx dy = \int_0^2 \left[ x \sin y^3 \right]_{x=0}^{x=y^2} dy \\
 &= \int_0^2 y^2 \sin y^3 dy = -\frac{1}{3} \cos y^3 \Big|_0^2 \\
 &= \frac{1 - \cos 8}{3}
 \end{aligned}$$



$$\begin{aligned}
 59. \int_0^4 \int_{\sqrt{y}}^2 \frac{1}{\sqrt{x^3+1}} dx dy &= \int_0^2 \int_0^{x^2} \frac{1}{\sqrt{x^3+1}} dy dx = \int_0^2 \left[ \frac{y}{\sqrt{x^3+1}} \right]_{y=0}^{y=x^2} dx \\
 &= \int_0^2 \frac{x^2}{\sqrt{x^3+1}} dx = \frac{2}{3} (x^3+1)^{1/2} \Big|_0^2 \\
 &= \frac{2}{3} (3-1) = \frac{4}{3}
 \end{aligned}$$

