

6. $F(x, y) = x^4 - x^2 + y^2 \Rightarrow \nabla F\left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right) = \left[(4x^3 - 2x)\mathbf{i} + 2y\mathbf{j}\right]_{(1/2, \sqrt{3}/4)} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$ is normal to the level curve $F(x, y) = x^4 - x^2 + y^2 = 0$ at $\left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right)$. So the slope of the required normal line is $m = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$ and an equation of the normal line is $y - \frac{\sqrt{3}}{4} = -\sqrt{3}\left(x - \frac{1}{2}\right) \Leftrightarrow y = -\sqrt{3}x + \frac{3\sqrt{3}}{4}$. The slope of the required tangent line is $m = -\frac{1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$, and so an equation of the tangent line is $y - \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{3}\left(x - \frac{1}{2}\right) \Leftrightarrow y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{12}$.
8. $F(x, y) = 2x + y - e^{x-y} \Rightarrow \nabla F(1, 1) = \left[(2 - e^{x-y})\mathbf{i} + (1 + e^{x-y})\mathbf{j}\right]_{(1,1)} = \mathbf{i} + 2\mathbf{j}$ is normal to the level curve $F(x, y) = 2x + y - e^{x-y} = 2$ at $(1, 1)$. From this, we see that the slope of the required normal line is $m = \frac{2}{1} = 2$ and so an equation of the normal line is $y - 1 = 2(x - 1) \Leftrightarrow y = 2x - 1$. The slope of the required tangent line is $m = -\frac{1}{2}$ and so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x - 1) \Leftrightarrow y = -\frac{1}{2}x + \frac{3}{2}$.
15. $F(x, y, z) = x^2 + 4y^2 + 9z^2 - 17 \Rightarrow \nabla F(2, 1, 1) = (2x\mathbf{i} + 8y\mathbf{j} + 18z\mathbf{k})|_{(2,1,1)} = 4\mathbf{i} + 8\mathbf{j} + 18\mathbf{k}$, so an equation of the tangent plane at $(2, 1, 1)$ is $4(x - 2) + 8(y - 1) + 18(z - 1) = 0 \Leftrightarrow 2x + 4y + 9z = 17$. Equations of the normal line passing through $(2, 1, 1)$ are $\frac{x-2}{4} = \frac{y-1}{8} = \frac{z-1}{18} \Leftrightarrow \frac{x-2}{2} = \frac{y-1}{4} = \frac{z-1}{9}$.
19. $F(x, y, z) = xy + yz + xz - 11 = 0 \Rightarrow \nabla F(1, 2, 3) = [(y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}]_{(1,2,3)} = 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$, so an equation of the tangent plane at $(1, 2, 3)$ is $5(x - 1) + 4(y - 2) + 3(z - 3) = 0 \Leftrightarrow 5x + 4y + 3z = 22$. Equations of the normal line passing through $(1, 2, 3)$ are $\frac{x-1}{5} = \frac{y-2}{4} = \frac{z-3}{3}$.
26. $F(x, y, z) = e^x \sin \pi y - z = 0 \Rightarrow \nabla F(0, 1, 0) = (e^x \sin \pi y \mathbf{i} + \pi e^x \cos \pi y \mathbf{j} - \mathbf{k})|_{(0,1,0)} = -\pi \mathbf{j} - \mathbf{k}$, so an equation of the tangent plane at $(0, 1, 0)$ is $-\pi(y - 1) - z = 0 \Leftrightarrow \pi y + z = \pi$. Equations of the normal line passing through $(0, 1, 0)$ are $x = 0, \frac{y-1}{-\pi} = \frac{z}{-1} \Leftrightarrow x = 0, \frac{y-1}{\pi} = z$.
28. $F(x, y, z) = \ln \frac{x}{y} - z = \ln x - \ln y - z = 0 \Rightarrow \nabla F(2, 2, 0) = \left(\frac{1}{x}\mathbf{i} - \frac{1}{y}\mathbf{j} - \mathbf{k}\right)|_{(2,2,0)} = \frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - \mathbf{k}$, so an equation of the tangent plane at $(2, 2, 0)$ is $\frac{1}{2}(x - 2) - \frac{1}{2}(y - 2) - z = 0 \Leftrightarrow x - y - 2z = 0$. Equations of the normal line passing through $(2, 2, 0)$ are $\frac{x-2}{1/2} = \frac{y-2}{-1/2} = \frac{z}{-1} \Leftrightarrow x - 2 = \frac{y-2}{-1} = \frac{z}{-2}$.
31. $F(x, y, z) = \sin xy + 3z - 3 = 0 \Rightarrow \nabla F(0, 3, 1) = (y \cos xy \mathbf{i} + x \cos xy \mathbf{j} + 3\mathbf{k})|_{(0,3,1)} = 3\mathbf{i} + 3\mathbf{k} = 3(\mathbf{i} + \mathbf{k})$, so an equation of the tangent plane at $(0, 3, 1)$ is $(x - 0) + (z - 1) = 0 \Leftrightarrow x + z = 1$. Equations of the normal line passing through $(0, 3, 1)$ are $\frac{x}{1} = \frac{z-1}{1}, y = 3$.

37. We write $F(x, y, z) = x^2 + y^2 + z^2 - 14$. Then the normal to the tangent plane to the sphere at the point (x_0, y_0, z_0) is $\nabla F(x_0, y_0, z_0) = 2x_0\mathbf{i} + 2y_0\mathbf{j} + 2z_0\mathbf{k}$. Since the tangent plane is parallel to the plane $x + 2y + 3z = 12$ whose normal is $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, we see that $x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} = c(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$, where c is a constant. This equation implies that $x_0 = c$, $y_0 = 2c$, and $z_0 = 3c$, and substituting these values into the equation of the sphere gives $c^2 + (2c)^2 + (3c)^2 = 14 \Leftrightarrow 14c^2 = 14 \Leftrightarrow c = \pm 1$. Thus, the required points are $(-1, -2, -3)$ and $(1, 2, 3)$.

13.8 Extrema of Functions of Two Variables

ET 12.8

3. To find the critical points of f , we solve the system

$$\left. \begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x}(-x^2 - 3y^2 + 4x - 6y + 8) = -2x + 4 = 0 \\ f_y(x, y) &= \frac{\partial}{\partial y}(-x^2 - 3y^2 + 4x - 6y + 8) = -6y - 6 = 0 \end{aligned} \right\} \text{obtaining the sole critical point } (2, -1) \text{ of } f.$$

Next, we use the SDT on the critical point: $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}^2(x, y) = (-2)(-6) - 0^2 = 12$. Since $D(2, -1) = 12 > 0$ and $f_{xx}(2, -1) = -2 < 0$, the point $(2, -1)$ gives a relative maximum of f with value $f(2, -1) = -2^2 - 3(-1)^2 + 4(2) - 6(-1) + 8 = 15$.

$$7. \left. \begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x}(2x^2 + y^2 - 2xy - 8x - 2y + 2) = 4x - 2y - 8 = 0 \\ f_y(x, y) &= \frac{\partial}{\partial y}(2x^2 + y^2 - 2xy - 8x - 2y + 2) = 2y - 2x - 2 = 0 \end{aligned} \right\} \Rightarrow x = 5, y = 6, \text{ so } (5, 6) \text{ is}$$

the sole critical point of f . Next, $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}^2(x, y) = 4 \cdot 2 - (-2)^2 = 4$. Since $D(5, 6) = 4 > 0$ and $f_{xx}(5, 6) = 4 > 0$, the point $(5, 6)$ gives a relative minimum of f with value $f(5, 6) = 2(5)^2 + 6^2 - 2(5)(6) - 8(5) - 2(6) + 2 = -24$.

$$9. \left. \begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x}(x^2 + 2y^2 + x^2y + 3) = 2x + 2xy = 0 \\ f_y(x, y) &= \frac{\partial}{\partial y}(x^2 + 2y^2 + x^2y + 3) = 4y + x^2 = 0 \end{aligned} \right\} \text{The first equation gives } x = 0 \text{ or}$$

$y = -1$. Substituting $x = 0$ into the second equation gives $y = 0$; substituting $y = -1$ into the second equation gives $x = \pm 2$. Thus, f has critical points $(0, 0)$, $(-2, -1)$, and $(2, -1)$. Next,

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}^2(x, y) = (2 + 2y)(4) - (2x)^2 = -4x^2 + 8y + 8.$$

At $(0, 0)$: $D(0, 0) = 8 > 0$ and $f_{xx}(0, 0) = 2 > 0$, so $(0, 0)$ gives a relative minimum of f with value $f(0, 0) = 3$.

At $(-2, -1)$: $D(-2, -1) = -4(-2)^2 + 8(-1) + 8 = -16 < 0$, so $(-2, -1, 5)$ is a saddle point of f .

At $(2, -1)$: $D(2, -1) = -4(2)^2 + 8(-1) + 8 = -16 < 0$, so $(2, -1, 5)$ is also a saddle point of f .

$$11. \left. \begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x}(x^2 + 5y^2 + x^2y + 2y^3) = 2x + 2xy = 2x(y + 1) = 0 \\ f_y(x, y) &= \frac{\partial}{\partial y}(x^2 + 5y^2 + x^2y + 2y^3) = 10y + x^2 + 6y^2 = 0 \end{aligned} \right\} \text{The first equation gives } x = 0 \text{ or } y = -1.$$

Substituting $x = 0$ into the second equation gives $10y + 6y^2 = 2y(3y + 5) = 0 \Rightarrow y = 0$ or $-\frac{5}{3}$; substituting $y = -1$ into the second equation gives $x^2 - 4 = 0 \Rightarrow x = \pm 2$. Thus, f has critical points $(0, 0)$, $(0, -\frac{5}{3})$, $(-2, -1)$, and $(2, -1)$. Next,

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}^2(x, y) = 2(y + 1)(10 + 12y) - (2x)^2 = 4(y + 1)(5 + 6y) - 4x^2.$$

At $(0, 0)$: $D(0, 0) = 4(1)(5) - 0 = 20 > 0$ and $f_{xx}(0, 0) = 2 > 0$, so f has a relative minimum at $(0, 0)$ with value $f(0, 0) = 0$.

At $(0, -\frac{5}{3})$: $D(0, -\frac{5}{3}) = 4(-\frac{5}{3} + 1)[5 + 6(-\frac{5}{3})] - 4(0) = \frac{40}{3} > 0$ and $f_{xx}(0, -\frac{5}{3}) = 2(-\frac{5}{3} + 1) = -\frac{4}{3} < 0$, so

f has a relative maximum at $(0, -\frac{5}{3})$ with value $f(0, -\frac{5}{3}) = \frac{125}{27}$.

At $(-2, -1)$: $D(-2, -1) = 0 - 4(-2)^2 = -16 < 0$, and so $(-2, -1, 3)$ is a saddle point of f .

At $(2, -1)$: $D(2, -1) = 0 - 4(2)^2 = -16 < 0$, and so $(2, -1, 3)$ is also a saddle point of f .

17.
$$\left. \begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x} (e^{-x^2-y^2}) = -2xe^{-x^2-y^2} \\ f_y(x, y) &= \frac{\partial}{\partial y} (e^{-x^2-y^2}) = -2ye^{-x^2-y^2} \end{aligned} \right\} \Rightarrow f_{xx}(x, y) = 2(2x^2 - 1)e^{-x^2-y^2}, f_{xy}(x, y) = 4xye^{-x^2-y^2}, \text{ and}$$

$$f_{yy}(x, y) = 2(2y^2 - 1)e^{-x^2-y^2}. \text{ Setting } \left. \begin{aligned} f_x(x, y) &= 0 \\ f_y(x, y) &= 0 \end{aligned} \right\} \Rightarrow x = 0 \text{ and } y = 0, \text{ so } (0, 0) \text{ is the sole critical point of } f.$$

Next, $D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}^2(0, 0) = (-2)(-2) - 0^2 = 4 > 0$ and $f_{xx}(0, 0) = -2 < 0$, we see that f has a relative maximum at $(0, 0)$ with value $f(0, 0) = 1$.

33. Since $f_x(x, y) = \frac{\partial}{\partial x} (2x + 3y - 6) = 2$ and

$f_y(x, y) = \frac{\partial}{\partial y} (2x + 3y - 6) = 3$ are never equal to 0, f has no critical point on D .

On $\ell_1, x = 2$ and $y = y$, so $g(y) = f(2, y) = 3y - 2$ for $-2 \leq y \leq 3$.

We see that g has an absolute minimum value of -8 at $(2, -2)$ and an absolute maximum value of 7 at $(2, 3)$.

On $\ell_2, x = x$ and $y = 3$, so $h(x) = f(x, 3) = 2x + 3$ for $0 \leq x \leq 2$. We see that h has an absolute minimum value of 3 at $(0, 3)$ and an absolute maximum value of 7 at $(2, 3)$.

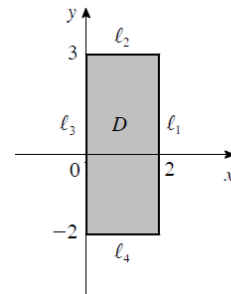
On $\ell_3, x = 0$ and $y = y$, so $s(y) = f(0, y) = 3y - 6$ for $-2 \leq y \leq 3$. We see that s has an absolute minimum value of -12 at $(0, -2)$ and an absolute maximum value of 3 at $(0, 3)$.

On $\ell_4, x = x$ and $y = -2$, so $t(x) = f(x, -2) = 2x - 12$ for $0 \leq x \leq 2$. We see that t has an absolute minimum value of -12 at $(0, -2)$ and an absolute maximum value of -8 at $(2, -2)$.

The extreme values of f on each boundary of D are summarized below.

	ℓ_1		ℓ_2		ℓ_3		ℓ_4	
(x, y)	$(2, -2)$	$(2, 3)$	$(0, 3)$	$(2, 3)$	$(0, -2)$	$(0, 3)$	$(0, -2)$	$(2, -2)$
Extreme value	-8	7	3	7	-12	3	-12	-8

We see that f has an absolute minimum value of $f(0, -2) = -12$ and an absolute maximum value of $f(2, 3) = 7$ on D .



35. $f_x(x, y) = 3$ and $f_y(x, y) = 4$, so f has no critical point.

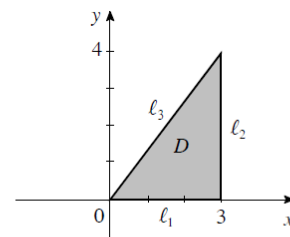
On $\ell_1, x = x$ and $y = 0$, so $g(x) = f(x, 0) = 3x - 12$ for $0 \leq x \leq 3$.

We see that f has an absolute minimum value of -12 and an absolute maximum value of -3 on ℓ_1 .

On $\ell_2, x = 3$ and $y = y$, so $h(y) = f(3, y) = 4y - 3$ for $0 \leq y \leq 4$. We see that f has an absolute minimum value of -3 and an absolute maximum value of 13 on ℓ_2 .

On $\ell_3, y = \frac{4}{3}x$, so $s(x) = f(x, \frac{4}{3}x) = 3x + 4(\frac{4}{3}x) - 12 = \frac{25}{3}x - 12$ for $0 \leq x \leq 3$. We see that f has an absolute minimum value of -12 and an absolute maximum value of 13 on ℓ_3 .

From these calculations, we see that f has an absolute minimum value of -12 and an absolute maximum value of 13 on D .



$$37. \left. \begin{aligned} f_x(x, y) &= \frac{\partial}{\partial x}(xy - x^2) = y - 2x = 0 \\ f_y(x, y) &= \frac{\partial}{\partial y}(xy - x^2) = x = 0 \end{aligned} \right\} \Rightarrow x = 0, y = 0, \text{ so } f \text{ has}$$

no critical point in the interior of D .

On C_1 , $y = x^2$, so $g(x) = f(x, x^2) = x^3 - x^2$ for $-2 \leq x \leq 2$.

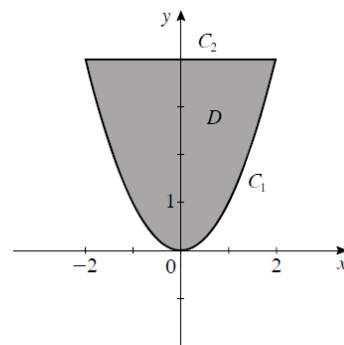
$g'(x) = 3x^2 - 2x = x(3x - 2) = 0 \Rightarrow x = 0$ or $x = \frac{2}{3}$, so 0 and $\frac{2}{3}$ are critical numbers of g on $(-2, 2)$.

From the table, we see that f has an absolute minimum value of -12 and an absolute maximum value of 4 on C_1 .

On C_2 , $x = x$ and $y = 4$, so $h(x) = f(x, 4) = 4x - x^2$ for $-2 \leq x \leq 2$.

$h'(x) = 4 - 2x = 0 \Rightarrow x = 2$, an endpoint. We find $h(-2) = -12$ and $g(2) = 4$, so f has an absolute minimum value of -12 and an absolute maximum value of 4 on C_2 .

We conclude that f has an absolute minimum value of -12 and an absolute maximum value of 4 on D .



x	-2	0	$\frac{2}{3}$	2
$g(x)$	-12	0	$-\frac{4}{27}$	4