

2.5 Derivatives of Trigonometric Functions

$$2. g'(x) = \frac{d}{dx}(x + \tan x) = 1 + \sec^2 x$$

$$4. y' = \frac{d}{dx}(\sqrt{x} \sin x) = x^{1/2} \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^{1/2}) = \sqrt{x} \cos x + \frac{\sin x}{2\sqrt{x}}$$

$$8. f'(t) = \frac{d}{dt}(\sec t \tan t) = \sec t \frac{d}{dt}(\tan t) + \tan t \frac{d}{dt}(\sec t) = (\sec t) \sec^2 t + \tan t (\sec t \tan t) = \sec t (\sec^2 t + \tan^2 t).$$

Alternative answers are $\sec t (1 + 2 \tan^2 t)$ or $\sec t (2 \sec^2 t - 1)$, using the identity $\sec^2 t = 1 + \tan^2 t$.

$$12. y' = \frac{d}{d\theta} \left(\frac{\cos \theta}{1 - \sin \theta} \right) = \frac{(1 - \sin \theta)(-\sin \theta) - \cos \theta(-\cos \theta)}{(1 - \sin \theta)^2} = \frac{-\sin \theta + \sin^2 \theta + \cos^2 \theta}{(1 - \sin \theta)^2} = \frac{1 - \sin \theta}{(1 - \sin \theta)^2} = \frac{1}{1 - \sin \theta}$$

$$21. f'(x) = \frac{d}{dx}(x \sin^2 x) = x \frac{d}{dx}[(\sin x)(\sin x)] + \sin^2 x \frac{d}{dx}(x) = x \left(\sin x \frac{d}{dx} \sin x + \sin x \frac{d}{dx} \sin x \right) + \sin^2 x \\ = 2x \sin x \cos x + \sin^2 x \quad \text{or} \quad x \sin 2x + \sin^2 x$$

$$24. g'(x) = \frac{d}{dx}(\sec x) = \sec x \tan x \Rightarrow$$

$$g''(x) = \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) = \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x = \sec x (\sec^2 x + \tan^2 x) \\ = \sec x (1 + 2 \tan^2 x)$$

$$26. h'(t) = \frac{d}{dt}[(t^2 + 1) \sin t] = (t^2 + 1) \frac{d}{dt}(\sin t) + \sin t \frac{d}{dt}(t^2 + 1) = (t^2 + 1) \cos t + 2t \sin t \Rightarrow$$

$$h''(t) = \frac{d}{dt}[(t^2 + 1) \cos t + 2t \sin t] = (t^2 + 1) \frac{d}{dt}(\cos t) + \cos t \frac{d}{dt}(t^2 + 1) + 2 \left[t \frac{d}{dt}(\sin t) + \sin t \frac{d}{dt}(t) \right] \\ = -(t^2 + 1) \sin t + 2t \cos t + 2t \cos t + 2 \sin t = 4t \cos t - t^2 \sin t + \sin t$$

$$46. \lim_{h \rightarrow 0} \frac{\tan\left(\frac{\pi}{4} + h\right) - 1}{h} = \frac{d}{dx}(\tan x) \Big|_{x=\pi/4} \quad (\text{by definition}) = \sec^2 x \Big|_{x=\pi/4} = (\sqrt{2})^2 = 2$$

2.6 The Chain Rule

2. $y = \sqrt{x^2 - 4} = g(f(x))$, where $u = f(x) = x^2 - 4$ and $y = g(u) = \sqrt{u} = u^{1/2}$, so

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{2}u^{-1/2} \cdot (2x) = \frac{x}{u^{1/2}} = \frac{x}{\sqrt{x^2 - 4}}.$$

6. $y = \sec \sqrt{x} = g(f(x))$, where $u = f(x) = \sqrt{x}$ and $y = g(u) = \sec u$, so

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (\sec u \tan u) \left(\frac{1}{2}x^{-1/2}\right) = \frac{\sec \sqrt{x} \tan \sqrt{x}}{2\sqrt{x}}.$$

12. $f(x) = \left(\frac{x^2 + 3}{x}\right)^{-2} \Rightarrow$

$$\begin{aligned} f'(x) &= -2 \left(\frac{x^2 + 3}{x}\right)^{-3} \frac{d}{dx} (x + 3x^{-1}) = -2 \left(\frac{x^2 + 3}{x}\right)^{-3} \left(1 - \frac{3}{x^2}\right) = -2 \left(\frac{x}{x^2 + 3}\right)^3 \left(\frac{x^2 - 3}{x^2}\right) \\ &= -\frac{2x(x^2 - 3)}{(x^2 + 3)^3} \end{aligned}$$

20. $f(x) = (x^2 + \sqrt{x})^6 = (x^2 + x^{1/2})^6 \Rightarrow f'(x) = 6(x^2 + x^{1/2})^5 (2x + \frac{1}{2}x^{-1/2}) = 6(x^2 + \sqrt{x})^5 \left(2x + \frac{1}{2\sqrt{x}}\right)$

24. $y = \frac{(t+1)^3}{(t^2+2t)^2} = (t+1)^3 (t^2+2t)^{-2} \Rightarrow$

$$\begin{aligned} \frac{dy}{dt} &= (t+1)^3 (-2) (t^2+2t)^{-3} (2t+2) + (t^2+2t)^{-2} (3)(t+1)^2 \\ &= (t+1)^2 (t^2+2t)^{-3} [-4(t+1)^2 + 3(t^2+2t)] = (t+1)^2 (t^2+2t)^{-3} (-4t^2 - 8t - 4 + 3t^2 + 6t) \\ &= -\frac{(t^2+2t+4)(t+1)^2}{t^3(t+2)^3} \end{aligned}$$

30. $y = \cos(x^3) \Rightarrow y' = -\sin(x^3) \frac{d}{dx}(x^3) = -3x^2 \sin(x^3)$

34. $f(x) = \tan^2 x + \cot(x^2) \Rightarrow f'(x) = 2 \tan x \sec^2 x - \csc^2(x^2) \frac{d}{dx}(x^2) = 2 \tan x \sec^2 x - 2x \csc^2(x^2)$

38. $g(x) = \tan^2(x^2 + x) \Rightarrow$

$$\begin{aligned} g'(x) &= 2 \tan(x^2 + x) \frac{d}{dx} \tan(x^2 + x) = 2 \tan(x^2 + x) \sec^2(x^2 + x) \frac{d}{dx}(x^2 + x) \\ &= 2(2x + 1) \tan(x^2 + x) \sec^2(x^2 + x) \end{aligned}$$

$$48. y = \sec^3 \left(\frac{\sqrt{x}}{1+x} \right) \Rightarrow$$

$$\begin{aligned} \frac{dy}{dx} &= 3 \sec^2 \left(\frac{\sqrt{x}}{1+x} \right) \sec \left(\frac{\sqrt{x}}{1+x} \right) \tan \left(\frac{\sqrt{x}}{1+x} \right) \frac{d}{dx} \left(\frac{\sqrt{x}}{1+x} \right) \\ &= 3 \sec^3 \left(\frac{\sqrt{x}}{1+x} \right) \tan \left(\frac{\sqrt{x}}{1+x} \right) \cdot \frac{(1+x) \frac{1}{2\sqrt{x}} - \sqrt{x}(1)}{(1+x)^2} = 3 \sec^3 \left(\frac{\sqrt{x}}{1+x} \right) \tan \left(\frac{\sqrt{x}}{1+x} \right) \cdot \frac{1+x-2x}{2\sqrt{x}(1+x)^2} \\ &= \frac{3(1-x)}{2\sqrt{x}(1+x)^2} \sec^3 \left(\frac{\sqrt{x}}{1+x} \right) \tan \left(\frac{\sqrt{x}}{1+x} \right) \end{aligned}$$

$$52. y = x \tan^2(2x+3) \Rightarrow$$

$$\begin{aligned} \frac{dy}{dx} &= x \frac{d}{dx} \tan^2(2x+3) + \tan^2(2x+3) \frac{d}{dx}(x) = x [2 \tan(2x+3)] \frac{d}{dx} [\tan(2x+3)] + \tan^2(2x+3) \\ &= 2x \tan(2x+3) \sec^2(2x+3) \frac{d}{dx}(2x+3) + \tan^2(2x+3) = \tan(2x+3) [4x \sec^2(2x+3) + \tan(2x+3)] \end{aligned}$$

$$66. F(x) = g(f(x)) \Rightarrow F'(x) = g'(f(x)) f'(x), \text{ so } F'(3) = g'(f(3)) f'(3) = g'(16) \cdot 6 = \frac{1}{8} \cdot 6 = \frac{3}{4}.$$

$$\begin{aligned} 76. g'(x) &= \frac{d}{dx} [x f(x^2+1)] = f(x^2+1) + x f'(x^2+1) \frac{d}{dx}(x^2+1) = f(x^2+1) + 2x^2 f'(x^2+1) \Rightarrow \\ g''(x) &= f'(x^2+1) \frac{d}{dx}(x^2+1) + 4x f'(x^2+1) + 2x^2 f''(x^2+1) \frac{d}{dx}(x^2+1) \\ &= 2x f'(x^2+1) + 4x f'(x^2+1) + 4x^3 f''(x^2+1) = 6x f'(x^2+1) + 4x^3 f''(x^2+1) \end{aligned}$$