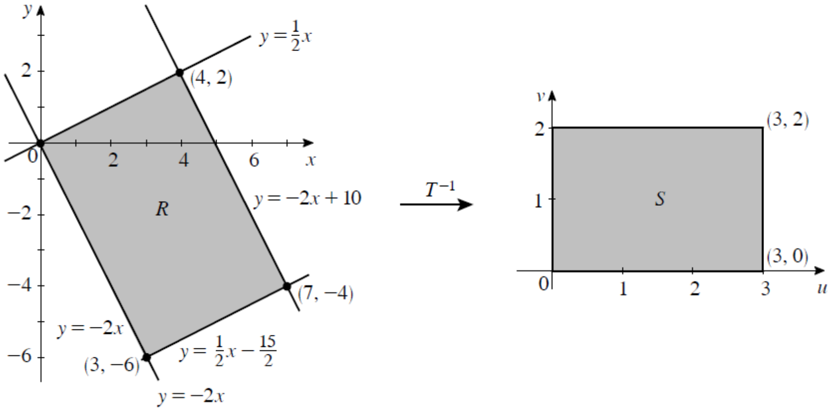
13. To find T^{-1}, we solve the system of equations of $T : x = u + 2v, y = v - 2u$ for u and v, obtaining $T^{-1} : u = \frac{1}{5}(x - 2y), v = \frac{1}{5}(2x + y)$. Using this transformation, we obtain the region $S = T^{-1}(R)$. Next, we find

$J = \dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 5$. Thus,

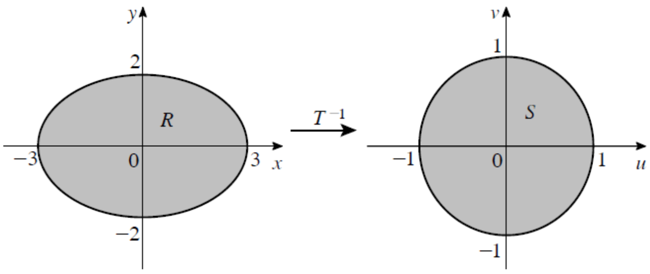
$$\iint_R (x + y)\, dA = \iint_S [(u + 2v) + (v - 2u)] \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv = 5 \int_0^2 \int_0^3 (3v - u)\, du\, dv$$

$$= 5 \int_0^2 \left[3uv - \tfrac{1}{2}u^2 \right]_{u=0}^{u=3} dv = 5 \int_0^2 \left(9v - \tfrac{9}{2} \right) dv = 5 \left(\tfrac{9}{2}v^2 - \tfrac{9}{2}v \right) \Big|_0^2 = 45$$



15. Here $T : x = 3u, y = 2v$. Then $4x^2 + 9y^2 = 36$ $\Rightarrow 4(3u)^2 + 9(2v)^2 = 36 \Leftrightarrow u^2 + v^2 = 1$, the circle with radius 1 centered at the origin of the uv-plane. Next, we find

$J = \dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6$, so



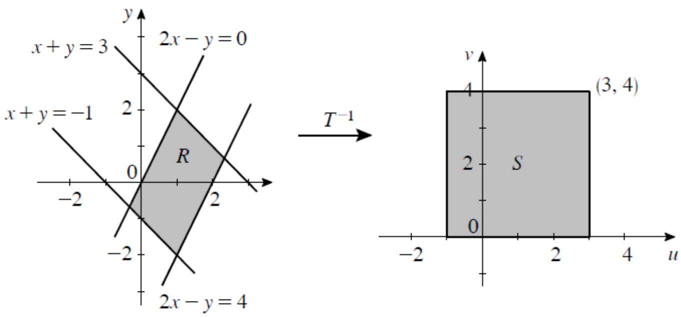
$$\iint_R 2xy\, dA = 2 \iint_S (3u)(2v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv = 12 \cdot 6 \int_0^{2\pi} \int_0^1 (r \cos\theta)(r \sin\theta)\, r\, dr\, d\theta = 72 \int_0^{2\pi} \left[\tfrac{1}{4}r^4 \cos\theta \sin\theta \right]_{r=0}^{r=1} d\theta$$

$$= 72 \int_0^{2\pi} \tfrac{1}{4} \cos\theta \sin\theta\, d\theta = -9 \sin^2\theta \Big|_0^{2\pi} = 0$$

21. Let $u = x + y$ and $v = 2x - y$. Then $-1 \le u \le 3$ and $0 \le v \le 4$. Solving for x and y, we obtain $x = \frac{1}{3}(u + v)$ and $y = \frac{1}{3}(2u - v)$.Thus,

$J = \dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{3}$,

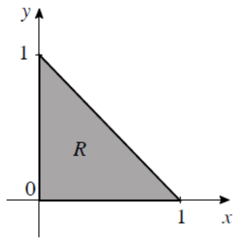
so

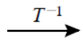


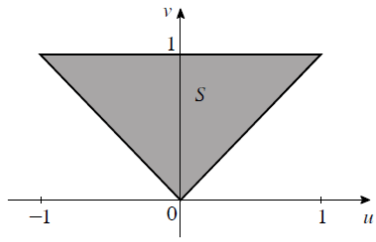
$$\iint_R (2x + y)\, dA = \int_{-1}^3 \int_0^4 \left[\tfrac{2}{3}(u + v) + \tfrac{1}{3}(2u - v) \right] \left| -\tfrac{1}{3} \right| dv\, du = \tfrac{1}{9} \int_{-1}^3 \int_0^4 (4u + v)\, dv\, du$$

$$= \tfrac{1}{9} \int_{-1}^3 \left[4uv + \tfrac{1}{2}v^2 \right]_{v=0}^{v=4} du = \tfrac{1}{9} \int_{-1}^3 (16u + 8)\, du = \tfrac{8}{9} \left(u^2 + u \right) \Big|_{-1}^3 = \tfrac{32}{3}$$

23. Let $T: u = x - y, v = x + y$. Then $T^{-1}: x = \frac{1}{2}(u + v), y = \frac{1}{2}(v - u)$. The triangular region R is mapped onto the triangular region S.





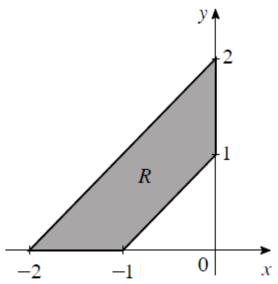


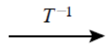
Here $J = \dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$, so

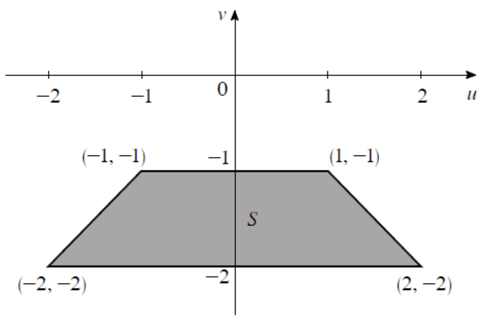
$$\iint_R \exp\left(\frac{x-y}{x+y}\right) dA = \int_0^1 \int_{-v}^{v} e^{u/v} \left|\tfrac{1}{2}\right| du\, dv = \tfrac{1}{2}\int_0^1 \left[v e^{u/v}\right]_{u=-v}^{u=v} dv = \tfrac{1}{2}\int_0^1 v\left(e - e^{-1}\right) dv = \tfrac{1}{2}\left(e - e^{-1}\right)\left(\tfrac{1}{2}v^2\right)\Big|_0^1$$

$$= \frac{e^2 - 1}{4e}$$

24. Let $T: u = x + y, v = x - y$, so $T^{-1}: x = \frac{1}{2}(u + v), y = \frac{1}{2}(u - v)$. The trapezoidal region R is mapped onto the trapezoidal region S.







Now $J = \dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$, so

$$\iint_R \exp\left(\frac{x+y}{x-y}\right) dA = \int_{-2}^{-1} \int_{-v}^{v} e^{u/v} \left|-\tfrac{1}{2}\right| du\, dv = \tfrac{1}{2}\int_{-2}^{-1} \left[v e^{u/v}\right]_{-v}^{v} dv = \tfrac{1}{2}\int_{-2}^{-1} \left(e^{-1} - e\right) v\, dv$$

$$= \frac{1 - e^2}{2e}\left(\frac{v^2}{2}\right)\Bigg|_{-2}^{-1} = \frac{3\left(e^2 - 1\right)}{4e}$$