7.1 Integration by Parts

2. $\int xe^{-x}\,dx$. Let $u = x$ and $dv = e^{-x}\,dx$. Then $du = dx$ and $v = \int e^{-x}\,dx = -e^{-x}$, so
$\int xe^{-x}\,dx = uv - \int v\,du = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C = -(x+1)\,e^{-x} + C$.

5. $\int x \ln 2x\,dx$. Let $u = \ln 2x$ and $dv = x\,dx$. Then $du = dx/x$ and $v = \int x\,dx = \frac{1}{2}x^2$, so
$\int x \ln 2x\,dx = uv - \int v\,du = \frac{1}{2}x^2 \ln 2x - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2 \ln 2x - \frac{1}{4}x^2 + C = \frac{1}{4}x^2\,(2\ln 2x - 1) + C$.

16. $I = \int e^{-x} \sin x\,dx$. Let $u = e^{-x}$ and $dv = \sin x\,dx$. Then $du = -e^{-x}\,dx$ and $v = \int \sin x\,dx = -\cos x$,
so $I = uv - \int v\,du = -e^{-x}\cos x - \int e^{-x}\cos x\,dx$. To find the integral on the right,
let $s = e^{-x}$ and $dt = \cos x\,dx$. Then $ds = -e^{-x}\,dx$ and $t = \int \cos x\,dx = \sin x$, so
$I = -e^{-x}\cos x - \left(e^{-x}\sin x + \int e^{-x}\sin x\,dx\right) = -e^{-x}\cos x - e^{-x}\sin x - I$. Then $2I = -e^{-x}(\cos x + \sin x)$ and
$I = -\frac{1}{2}e^{-x}(\cos x + \sin x) + C$.

19. $\int u \sin(2u+1)\,du$. Let $p = u$ and $dq = \sin(2u+1)\,du$, so $dp = du$, $q = \int \sin(2u+1)\,du = -\frac{1}{2}\cos(2u+1)$, and

$$\int u\sin(2u+1)\,du = pq - \int q\,dp = -\tfrac{1}{2}u\cos(2u+1) + \tfrac{1}{2}\int \cos(2u+1)\,du$$
$$= -\tfrac{1}{2}u\cos(2u+1) + \tfrac{1}{4}\sin(2u+1) + C = \tfrac{1}{4}\left[\sin(2u+1) - 2u\cos(2u+1)\right] + C$$

34. $\int_0^2 \ln(x+1)\,dx$. Let $u = x+1$, so $du = dx$, $x = 0 \Rightarrow u = 1$, and $x = 2 \Rightarrow u = 3$. Then
$\int_0^2 \ln(x+1)\,dx = \int_1^3 \ln u\,du = (u\ln u - u)|_1^3 = 3\ln 3 - 2$. (See Example 6.)

36. $\int_0^\pi x\sin 2x\,dx$. Let $u = x$ and $dv = \sin 2x\,dx$, so $du = dx$ and $v = -\frac{1}{2}\cos 2x$. Then
$\int_0^\pi x\sin 2x\,dx = -\frac{1}{2}x\cos 2x\Big|_0^\pi + \frac{1}{2}\int_0^\pi \cos 2x\,dx = -\frac{\pi}{2} + \left[\frac{1}{4}\sin 2x\right]_0^\pi = -\frac{\pi}{2}$.

37. $\int_{\sqrt{e}}^{e} x^{-2}\ln x\,dx$. Let $u = \ln x$ and $dv = x^{-2}\,dx$, so $du = dx/x$ and $v = -1/x$. Then

$$\int_{\sqrt{e}}^{e} x^{-2}\ln x\,dx = -\frac{\ln x}{x}\Big|_{\sqrt{e}}^{e} + \int_{\sqrt{e}}^{e} x^{-2}\,dx = -\frac{1}{e} + \frac{\ln\sqrt{e}}{\sqrt{e}} - \left[\frac{1}{x}\right]_{\sqrt{e}}^{e} = -\frac{1}{e} + \frac{1}{2\sqrt{e}} - \frac{1}{e} + \frac{1}{\sqrt{e}} = \frac{3}{2\sqrt{e}} - \frac{2}{e} = \frac{3\sqrt{e}-4}{2e}.$$

38. $I = \int_0^{\pi/2} e^{2x}\cos x\,dx$. Let $u = e^{2x}$ and $dv = \cos x\,dx$, so $du = 2e^{2x}\,dx$ and $v = \sin x$. Then
$\int_0^{\pi/2} e^{2x}\cos x\,dx = e^{2x}\sin x\Big|_0^{\pi/2} - 2\int_0^{\pi/2} e^{2x}\sin x\,dx = e^{\pi} - 2\int_0^{\pi/2} e^{2x}\sin x\,dx$. Now let $s = e^{2x}$ and $dt = \sin x\,dx$,
so $ds = 2e^{2x}\,dx$ and $t = -\cos x$. Then $I = e^{\pi} - 2\left\{\left[-e^{2x}\cos x\right]_0^{\pi/2} + 2\int_0^{\pi/2} e^{2x}\cos x\,dx\right\} = e^{\pi} - 2 - 4I \Leftrightarrow$
$5I = e^{\pi} - 2 \Leftrightarrow I = \int_0^{\pi/2} e^{2x}\cos x\,dx = \frac{1}{5}(e^{\pi} - 2)$.

67. $\int_1^3 xf''(x)\,dx$. Let $u = x$ and $dv = f''(x)\,dx$, so $du = dx$ and $v = f'(x)$. Then

$$\int_1^3 xf''(x)\,dx = xf'(x)\Big|_1^3 - \int_1^3 f'(x)\,dx = 3f'(3) - f'(1) - \left[f(x)\right]_1^3 = 3f'(3) - f'(1) - f(3) + f(1)$$
$$= 3(5) - 2 - (-1) + 2 = 16$$

68. Letting $u = \frac{1}{x}$ and $dv = dx$, so $du = -\frac{dx}{x^2}$ and $v = x + C_1$, the integration by parts formula gives

$$\int \frac{dx}{x} = uv - \int v\,du = \frac{1}{x}(x + C_1) - \int (x + C_1)\left(-\frac{1}{x^2}\right)dx = 1 + \frac{C_1}{x} + \int \frac{dx}{x} + \int C_1 x^{-2}\,dx$$

$$= 1 + \frac{C_1}{x} + \int \frac{dx}{x} + C_1\left(-\frac{1}{x} + C_2\right)$$

or $0 = 1 + C_1 C_2$. Because C_1 and C_2 are arbitrary constants, this is a contradiction, and therefore the so-called proof is invalid.