1.4 Continuous Functions

12. The denominator of the function f is equal to 0 when $(x-3)(x+1)=0$, that is, when $x=3$ or $x=-1$. So f is discontinuous at 3 and -1.

24. Since $\lim_{x\to 0}(-|x|+1)=1$ and $f(0)=0$, f is discontinuous at 0.

30. We require that $\lim_{x\to 2^-}(kx+1)=\lim_{x\to 2^+}\left(kx^2-3\right)$, or $2k+1=4k-3$. Solving this last equation, we obtain $k=2$.

31. Since $\lim_{x\to 0}\frac{\sin 2x}{x}=\lim_{x\to 0}\left(2\cdot\frac{\sin 2x}{2x}\right)=2$, we require that $f(0)=2$. So we take $c=2$.

52. $\lim_{x\to 1}\frac{2x^3+x-3}{x-1}=\lim_{x\to 1}\frac{(x-1)\left(2x^2+2x+3\right)}{x-1}=\lim_{x\to 1}\left(2x^2+2x+3\right)=7$. So f will be continuous at 1 if we define

$$f(x)=\begin{cases}\dfrac{2x^3+x-3}{x-1} & \text{if } x\neq 1\\ 7 & \text{if } x=1\end{cases}$$

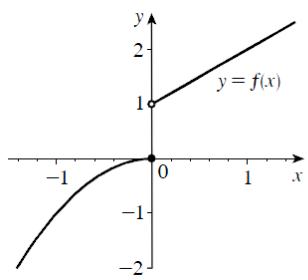
54. $\lim_{x\to 4}\frac{4-x}{2-\sqrt{x}}=\lim_{x\to 4}\frac{(2-\sqrt{x})(2+\sqrt{x})}{2-\sqrt{x}}=\lim_{x\to 4}(2+\sqrt{x})=4$. So we define $f(x)=\begin{cases}\dfrac{4-x}{2-\sqrt{x}} & \text{if } x\neq 4\\ 4 & \text{if } x=4\end{cases}$

60. $f(x)=x^2-4x+6$ is continuous on $[0,3]$. $f(0)=6$ and $f(3)=3$. Since $f(3)\leq 3\leq f(0)$, there exists a number c in $[0,3]$ such that $f(c)=3$. To find c we solve $x^2-4x+6=3\Rightarrow x^2-4x+3=(x-3)(x-1)=0$ giving $x=1$ or 3. Therefore $c=1$ or 3.

64. $f(x)=x^4-2x^3-3x^2+7$ is continuous on $[1,2]$. $f(1)=3>0$ and $f(2)=-5<0$. Therefore, by Theorem 7, $f(x)=0$ has at least one root in $(1,2)$.

70. a. $\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}\left(-x^2\right)=0$ and $\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}(x+1)=1$. This shows that f is not continuous at 0, and therefore is not continuous on $[-1,1]$.

b. $f(-1)=-1$ and $f(1)=2$. The number $\frac{1}{2}$ lies between $f(-1)$ and $f(1)$, but there is no c in $[-1,1]$ such that $f(c)=\frac{1}{2}$. (See the figure.)



1.5 Tangent Lines and Rates of Change

12. a. $m_{\text{sec}} = \dfrac{f(2+h) - f(2)}{(2+h) - 2} = \dfrac{\left[(2+h)^3 + (2+h)\right] - \left(2^3 + 2\right)}{h} = \dfrac{h^3 + 6h^2 + 13h}{h} = h^2 + 6h + 13$

b. $m_{\tan} = \lim\limits_{h\to 0} \dfrac{f(2+h) - f(2)}{(2+h) - 2} = \lim\limits_{h\to 0} \left(h^2 + 6h + 13\right) = 13$

c. $y - 10 = 13(x - 2) \Rightarrow y = 13x - 16$

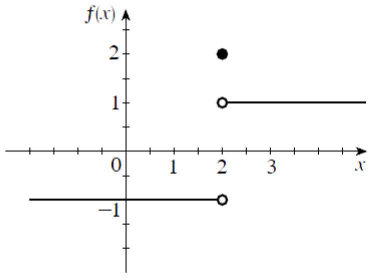
18. $\lim\limits_{h\to 0} \dfrac{f(4+h) - f(4)}{(4+h) - 4} = \lim\limits_{h\to 0} \dfrac{\sqrt{4+h} - \sqrt{4}}{h} = \lim\limits_{h\to 0} \dfrac{\left(\sqrt{4+h} - 2\right)\left(\sqrt{4+h} + 2\right)}{h\left(\sqrt{4+h} + 2\right)} = \lim\limits_{h\to 0} \dfrac{h}{h\left(\sqrt{4+h} + 2\right)} = \dfrac{1}{4}$

36. Using the definition of the derivative, we find $f(x) = 2x^{1/4}$ and $a = 16$.

38. Using the definition of the derivative, we find $f(x) = 2^x$ and $a = 3$.

Chapter 1 Review

4.



a. $\lim\limits_{x\to 2^-} f(x) = -1$

b. $\lim\limits_{x\to 2^+} f(x) = 1$

c. $\lim\limits_{x\to 2} f(x)$ does not exist.

12. $\lim\limits_{x\to 3} \dfrac{27 - x^3}{x - 3} = \lim\limits_{x\to 3} \dfrac{-(x-3)\left(x^2 + 3x + 9\right)}{x - 3} = \lim\limits_{x\to 3} \left[-\left(x^2 + 3x + 9\right)\right] = -27$

16. $\lim\limits_{x\to 4} \dfrac{x - 4}{\sqrt{x} - 2} = \lim\limits_{x\to 4} \dfrac{\left(\sqrt{x} + 2\right)\left(\sqrt{x} - 2\right)}{\sqrt{x} - 2} = \lim\limits_{x\to 4} \left(\sqrt{x} + 2\right) = 4$

18. $\lim\limits_{x\to 2^+} \dfrac{x - 2}{|x - 2|} = \lim\limits_{x\to 2^+} \dfrac{x - 2}{x - 2} = 1$ since $|x - 2| = x - 2$ for all $x \geq 2$.

22. $\lim\limits_{x\to 0} x \cot 2x = \lim\limits_{x\to 0} x \cdot \dfrac{\cos 2x}{\sin 2x} = \lim\limits_{x\to 0} \cos 2x \cdot \lim\limits_{x\to 0} \dfrac{\frac{1}{2}(2x)}{\sin 2x} = 1 \cdot \dfrac{1}{2} \cdot \lim\limits_{x\to 0} \dfrac{2x}{\sin 2x} = \dfrac{1}{2}$

42'. If $x \neq 0$, then $f(x)$ is continuous. Next, observe that $-|x| \leq x \sin(1/x) \leq |x|$. Since $|\sin(1/x)| \leq 1$, the Squeeze Theorem implies that $\lim_{x\to 0} x \sin(1/x) = 0 = f(0)$, and so f is continuous at 0 as well. Therefore, f is continuous on $(-\infty, \infty)$.

2.1 The Derivative

12. $f(x) = -\frac{2}{\sqrt{x}} \quad \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{-\frac{2}{\sqrt{x+h}} - \left(-\frac{2}{\sqrt{x}}\right)}{h} = 2 \lim_{h\to 0} \frac{\frac{-\sqrt{x} + \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}}{h}$$

$$= 2 \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h\sqrt{x}\sqrt{x+h}} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = 2 \lim_{h\to 0} \frac{(x+h) - x}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= 2 \lim_{h\to 0} \frac{h}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x+h} + \sqrt{x}\right)} = 2 \lim_{h\to 0} \frac{1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{2}{\sqrt{x}\sqrt{x}\left(2\sqrt{x}\right)}$$

$$= \frac{1}{x\sqrt{x}} \text{ with domain } (0, \infty).$$

15. $f(x) = x^2 + 1 \quad \Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left[(x+h)^2 + 1\right] - \left(x^2 + 1\right)}{h} = \lim_{h\to 0} \frac{\left(x^2 + 2xh + h^2 + 1\right) - \left(x^2 + 1\right)}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} (2x + h) = 2x$$

The slope of the tangent line at $(2, 5)$ is $f'(2) = 2(2) = 4$. An equation of the tangent line is $y - 5 = 4(x - 2)$ or $y = 4x - 3$.

50. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x+1) = 1$, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(x^2 + 1\right) = 1$. Therefore, $\lim_{x\to 0} f(x) = 1$. Also, $f(0) = 0 + 1 = 1$, and so $\lim_{x\to 0} f(x) = f(0)$. Therefore, f is continuous at 0.

To show that f is not differentiable at 0, let $h < 0$ and consider $\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{(h+1) - 1}{h} = \lim_{h\to 0^-} 1 = 1$. Next, if $h > 0$, then $\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{\left[(0+h)^2 + 1\right] - 1}{h} = \lim_{h\to 0^+} h = 0$. This shows that $\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$ does not exist, and so by definition, f is not differentiable at 0.

52. $f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ Because $|\sin(1/x)| \leq 1$, we have $0 \leq |x\sin(1/x)| \leq |x|$. Since $\lim_{x\to 0} |x| = 0$, the Squeeze Theorem implies that $\lim_{x\to 0} x\sin(1/x) = 0 = f(0)$, and so f is continuous at 0.

To show that f is not differentiable at 0, we compute $f'(0) = \lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \dfrac{h\sin(1/h)}{h} = \lim_{h\to 0} \sin(1/h)$ which does not exist. Therefore, f is not differentiable at 0.