

4.3 Area

$$40. \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{n^3} (2k+1)^2 = \lim_{n \rightarrow \infty} \frac{1}{n^3} \left[4 \sum_{k=1}^n k^2 + 4 \sum_{k=1}^n k + \sum_{k=1}^n 1 \right] = \lim_{n \rightarrow \infty} \frac{1}{n^3} \left[\frac{4n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} + n \right]$$

$$= \lim_{n \rightarrow \infty} \frac{2}{3} \left[\left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) + 2 \left(\frac{1}{n} + \frac{1}{n^2}\right) + \frac{1}{n^2} \right] = \frac{4}{3}$$

$$42. \lim_{n \rightarrow \infty} \sum_{k=1}^n \left[1 + 2 \left(\frac{k}{n}\right)^2 \right] \left(\frac{2}{n}\right) = \lim_{n \rightarrow \infty} \left(\frac{2}{n}\right) \left[\sum_{k=1}^n 1 + \frac{2}{n^2} \sum_{k=1}^n k^2 \right] = \lim_{n \rightarrow \infty} \left[\left(\frac{2}{n}\right)n + \frac{2}{n} \cdot \frac{2}{n^2} \cdot \frac{n(n+1)(2n+1)}{6} \right]$$

$$= \lim_{n \rightarrow \infty} \left[2 + \frac{2}{3} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) \right] = 2 + \frac{4}{3} = \frac{10}{3}$$

$$44. \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(1 + \frac{2k-1}{2n}\right) \left(\frac{1}{n}\right) = \lim_{n \rightarrow \infty} \frac{1}{n} \left[\sum_{k=1}^n \frac{2n+2k-1}{2n} \right] = \lim_{n \rightarrow \infty} \frac{1}{2n^2} \left[(2n-1) \sum_{k=1}^n 1 + 2 \sum_{k=1}^n k \right]$$

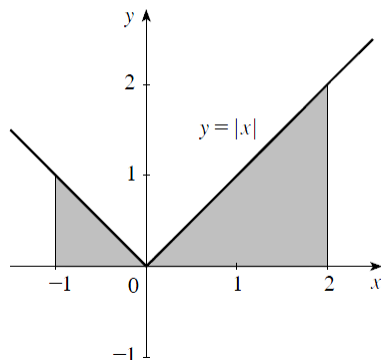
$$= \lim_{n \rightarrow \infty} \left[\frac{2n-1}{2n^2} \cdot n + \frac{1}{n^2} \cdot \frac{n(n+1)}{2} \right] = \lim_{n \rightarrow \infty} \left[1 - \frac{1}{2n} + \frac{1}{2} \left(1 + \frac{1}{n}\right) \right] = \frac{3}{2}$$

4.4 The Definite Integral

$$15. \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{2c_k}{c_k^2 + 1} \Delta x = \int_1^2 \frac{2x}{x^2 + 1} dx$$

$$16. \lim_{n \rightarrow \infty} \sum_{k=1}^n c_k \cos c_k \Delta x = \int_0^{\pi/2} x \cos x dx$$

$$22. \int_{-1}^2 |x| dx = \frac{1}{2} (1) (1) + \frac{1}{2} (2) (2) = \frac{5}{2}$$



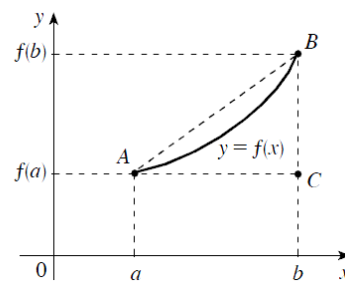
$$48. -|f(x)| \leq f(x) \leq |f(x)| \text{ for all } x \text{ in } [a, b], \text{ so by Property 5 of the definite integral,}$$

$$-\int_a^b |f(x)| dx = \int_a^b [-|f(x)|] dx \leq \int_a^b f(x) dx \leq \int_a^b |f(x)| dx \Rightarrow \left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx.$$

50. Without loss of generality, we may assume that $f(x) > 0$ on $[a, b]$. Since f is concave upward on $[a, b]$, the graph of f lies below the chord connecting A and B . Therefore,

$$\left[\begin{array}{c} \text{area of} \\ \text{rectangle } abCA \end{array} \right] \leq \left[\begin{array}{c} \text{area under} \\ \text{the graph of } f \end{array} \right] \leq \left[\begin{array}{c} \text{area of} \\ \text{trapezoid } abBA \end{array} \right] \Rightarrow$$

$$(b-a)f(a) \leq \int_a^b f(x) dx \leq \frac{1}{2}(b-a)[f(a) + f(b)].$$



4.5 The Fundamental Theorem of Calculus

4. $G'(x) = \frac{d}{dx} \int_{-1}^x t\sqrt{t^2+1} dt = x\sqrt{x^2+1}$

6. $h'(x) = \frac{d}{dx} \int_x^3 \frac{t}{\sqrt{t+1}} dt = -\frac{d}{dx} \int_3^x \frac{t}{\sqrt{t+1}} dt = -\frac{x}{\sqrt{x+1}}$

8. $G'(x) = \frac{d}{dx} \int_0^{x^2} t \sin t dt = (x^2 \sin x^2) \frac{d}{dx} (x^2) = 2x^3 \sin x^2$

11. $F'(x) = \frac{d}{dx} \int_1^{\cos x} \frac{t^2}{t+1} dt = \frac{\cos^2 x}{\cos+1} \cdot \frac{d}{dx} (\cos x) = -\frac{\sin x \cos^2 x}{\cos x + 1}$

12. $G'(x) = \frac{d}{dx} \int_{\sqrt{x}}^5 \frac{\sin t^2}{t} dt = -\frac{d}{dx} \int_5^{x^{1/2}} \frac{\sin t^2}{t} dt = -\frac{\sin x}{\sqrt{x}} \cdot \frac{d}{dx} (x^{1/2}) = -\frac{\sin x}{2x}$

20. $\int_1^2 \frac{3x^4 - 2x^2 + 1}{2x^2} dx = \frac{1}{2} \int_1^2 (3x^2 - 2 + x^{-2}) dx = \frac{1}{2} \left(x^3 - 2x - \frac{1}{x} \right) \Big|_1^2 = \frac{1}{2} \left[\left(8 - 4 - \frac{1}{2} \right) - \left(1 - 2 - 1 \right) \right] = \frac{11}{4}$

21. $\int_4^9 \frac{x-1}{\sqrt{x}} dx = \int_4^9 (x^{1/2} - x^{-1/2}) dx = \frac{2}{3}x^{3/2} - 2x^{1/2} \Big|_4^9 = \frac{2}{3}\sqrt{x}(x-3) \Big|_4^9 = \frac{2}{3} [\sqrt{9}(9-3) - \sqrt{4}(4-3)] = \frac{32}{3}$

25. $\int_{\pi/6}^{\pi/4} \sec^2 t dt = \tan t \Big|_{\pi/6}^{\pi/4} = \tan \frac{\pi}{4} - \tan \frac{\pi}{6} = 1 - \frac{\sqrt{3}}{3} = \frac{3-\sqrt{3}}{3}$

27. $\int_0^{\pi} \sin 2x \cos x dx = \int_0^{\pi} (2 \sin x \cos x) \cos x dx = 2 \int_0^{\pi} \cos^2 x \sin x dx = -\frac{2}{3} \cos^3 x \Big|_0^{\pi} = -\frac{2}{3} (-1 - 1) = \frac{4}{3}$

28. $\int_0^{\pi} |\cos x| dx = \int_0^{\pi/2} \cos x dx - \int_{\pi/2}^{\pi} \cos x dx = [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi} = (1 - 0) - (0 - 1) = 2$

35. Let $u = t^2 - 1$, so $du = 2t dt$, $t = 1 \Rightarrow u = 0$, and $t = 2 \Rightarrow u = 3$. Then

$$\int_1^2 8t (t^2 - 1)^7 dt = 4 \int_0^3 u^7 du = \frac{1}{2} u^8 \Big|_0^3 = \frac{1}{2} (3^8) = \frac{6561}{2}.$$

37. Let $v = 5 - u$, so $dv = -du$, $u = 1 \Rightarrow v = 4$, and $u = 4 \Rightarrow v = 1$. Then

$$\int_1^4 \sqrt[3]{5-u} du = - \int_4^1 v^{1/3} dv = \int_1^4 v^{1/3} dv = \frac{3}{4} v^{4/3} \Big|_1^4 = \frac{3}{4} (4\sqrt[3]{4} - 1).$$

39. Let $u = \sqrt{x} + 1$, so $du = \frac{1}{2}x^{-1/2} dx = \frac{1}{2\sqrt{x}} dx \Rightarrow \frac{dx}{\sqrt{x}} = 2 du$, $x = 1 \Rightarrow u = 2$, and $x = 4 \Rightarrow u = 3$. Thus,

$$\int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)^2} dx = 2 \int_2^3 \frac{du}{u^2} = 2 \int_2^3 u^{-2} du = -\frac{2}{u} \Big|_2^3 = -\frac{2}{3} + 1 = \frac{1}{3}.$$

42. Let $u = \cos \theta$, so $du = -\sin \theta d\theta \Rightarrow \sin \theta d\theta = -du$, $\theta = 0 \Rightarrow u = 1$, and $\theta = \frac{\pi}{2} \Rightarrow u = 0$. Then

$$\int_0^{\pi/2} \sqrt{\cos \theta} \sin \theta d\theta = - \int_1^0 u^{1/2} du = \int_0^1 u^{1/2} du = \frac{2}{3} u^{3/2} \Big|_0^1 = \frac{2}{3}.$$

60. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^{1/3} = \int_0^1 x^{1/3} dx = \frac{3}{4} x^{4/3} \Big|_0^1 = \frac{3}{4}$

62. $\lim_{n \rightarrow \infty} \frac{\pi}{2n} \sum_{k=1}^n \cos\left(\frac{k\pi}{2n}\right) = \int_0^{\pi/2} \cos x dx = \sin x \Big|_0^{\pi/2} = \sin \frac{\pi}{2} - \sin 0 = 1$

88. $\frac{d}{dy} \left[\int_0^x \sqrt{3+2\cos t} dt + \int_0^y \sin t dt \right] = \frac{d}{dy} (0) = 0 \Rightarrow \sqrt{3+2\cos x} \frac{dx}{dy} + \sin y = 0 \Rightarrow \frac{dx}{dy} = -\frac{\sin y}{\sqrt{3+2\cos x}}$

92. Let $F(x) = \int_2^x \sqrt{5+t^2} dt$. Then

$$F'(2) = \lim_{h \rightarrow 0} \frac{F(2+h) - F(2)}{h} = \lim_{h \rightarrow 0} \frac{\int_0^{2+h} \sqrt{5+t^2} dt - \int_0^2 \sqrt{5+t^2} dt}{h} = \lim_{h \rightarrow 0} \frac{\int_2^{2+h} \sqrt{5+t^2} dt}{h}.$$

Next, using FTC1,

we find $F'(x) = \frac{d}{dx} \int_2^x \sqrt{5+t^2} dt = \sqrt{5+x^2} \Rightarrow F'(2) = \sqrt{5+2^2} = 3$, so we have $\lim_{h \rightarrow 0} \frac{1}{h} \int_2^{2+h} \sqrt{5+t^2} dt = 3$.