

9.7 Power Series

5. Let $u_n = \frac{(2x)^n}{n!}$. Then $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{(n+1)!} \cdot \frac{n!}{(2x)^n} \right| = \lim_{n \rightarrow \infty} \left(\frac{2}{n+1} \right) |x| = 0$ for any real x , so the radius of convergence is infinite and the interval of convergence is $(-\infty, \infty)$.

8. Let $u_n = \frac{n!x^n}{(2n)!}$. Then $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)!x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n!x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{2(2n+1)} = 0$ for any real x , so the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

21. Let $u_n = \frac{2^n(x+2)^n}{n^n}$. Then

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}(x+2)^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n(x+2)^n} \right| = \lim_{n \rightarrow \infty} \left(\frac{2}{n+1} \right) \left(\frac{n}{n+1} \right)^n |x+2| = 0 \text{ for any real } x \text{ since}$$

$$\lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^{-n} = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n} \right) \right]^{-n} = \frac{1}{e}. \text{ Thus, } R = \infty \text{ and the interval of convergence is } (-\infty, \infty).$$

25. Let $u_n = \frac{x^n}{n(\ln n)^2}$. Then

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)[\ln(n+1)]^2} \cdot \frac{n(\ln n)^2}{x^n} \right| = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right) \left[\frac{\ln n}{\ln(n+1)} \right]^2 |x| = |x|, \text{ so } R = 1 \text{ and the}$$

series converges on $(-1, 1)$. At $x = -1$ the series is $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$, which converges, and at $x = 1$ it is $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$, which also converges (see Exercise 9.3.33). Thus, the interval of convergence is $[-1, 1]$.

9.8 Taylor and Maclaurin Series

3. $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$, ..., $f^{(n)}(x) = e^x$, ...
 $f(2) = e^2$, $f'(2) = e^2$, $f''(2) = e^2$, ..., $f^{(n)}(2) = e^2$, ...

The required Taylor series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n = e^2 \left[1 + (x-2) + \frac{1}{2!} (x-2)^2 + \cdots + \frac{1}{n!} (x-2)^n + \cdots \right].$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{e^2(x-2)^{n+1}}{(n+1)!} \cdot \frac{n!}{e^2(x-2)^n} \right| = \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} \right) |x-2| = 0, \text{ so } R = \infty.$$

8. $f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$, $f'''(x) = -\frac{3 \cdot 2}{x^4}$, ..., $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$, ...
 $f(-1) = -1$, $f'(-1) = -1$, $f''(-1) = -2$, $f'''(-1) = -3 \cdot 2$, ..., $f^{(n)}(-1) = -n!$, ...

The required Taylor series is $\sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!} (x+1)^n = \sum_{n=0}^{\infty} \frac{-n!}{n!} (x+1)^n = -\sum_{n=0}^{\infty} (x+1)^n$.

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x+1)^{n+1}}{(x+1)^n} \right| = |x+1|, \text{ so } R = 1.$$

16. $f(x) = \frac{1}{4+x^2} = \frac{1}{4} \cdot \frac{1}{1+(\frac{x}{2})^2} = \frac{1}{4} \cdot \frac{1}{1-[-(\frac{x}{2})^2]} = \frac{1}{4} \sum_{n=0}^{\infty} \left[-\left(\frac{x}{2}\right)^2 \right]^n = \frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} x^{2n}$. The

series converges if $\left| \frac{x}{2} \right| < 1$ or $|x| < 2$, so $R = 2$.

21. $f(x) = \cos^2 x = \frac{1}{2}(1 + \cos 2x) = \frac{1}{2} + \frac{1}{2} \cos 2x = \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}$ with $R = \infty$.