9.5 Alternating Series

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is an alternating series with $a_n = \frac{1}{\sqrt{n}}$. Since $a_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = a_n$ and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$, the AST implies that the given series converges.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+2} = 0$, the AST implies that the given series converges.

9. $\sum_{n=2}^{\infty} \frac{(-1)^n n}{\ln n}$ is an alternating series with $a_n = \frac{n}{\ln n}$, but $\lim_{n\to\infty} \frac{(-1)^n n}{\ln n}$ does not exist because, using l'Hôpital's Rule,

$\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty$. Thus, the series diverges by the Divergence Test.

17. $\sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{\sqrt{n^3+1}} = \frac{1}{\sqrt{1^3+1}} - \frac{1}{\sqrt{3^3+1}} + \frac{1}{\sqrt{5^3+1}} - \cdots$ is an alternating series. Since $\left\{ \frac{1}{\sqrt{(2k+1)^3+1}} \right\}$ is decreasing and $\lim_{k\to\infty} \frac{1}{\sqrt{(2k+1)^3+1}} = 0$, the AST implies that the given series converges.

19. $\sum_{n=1}^{\infty} (-1)^n n \sin \frac{\pi}{n}$. Since $\lim_{n\to\infty} n \sin \frac{\pi}{n} = \lim_{n\to\infty} \pi \cdot \frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}} = \pi$, we see that $\lim_{n\to\infty} (-1)^n n \sin \frac{\pi}{n}$ does not exist, so the series diverges by the Divergence Test.

9.6 Absolute Convergence: The Ratio and Root Tests

3. $\sum_{n=1}^{\infty} \frac{(-2)^{n-1}}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{n-1}}{n^2}$. Since $\lim_{x\to\infty} \frac{2^{x-1}}{x^2} = \lim_{x\to\infty} \frac{2^{x-1} \ln 2}{2x} = \lim_{x\to\infty} \frac{2^{x-1} (\ln 2)^2}{2} = \infty$, we see that $\lim_{n\to\infty} a_n \neq 0$, so the series diverges by the Divergence Test.

11. $\sum_{n=1}^{\infty} \frac{n!}{e^n}$. Using the Ratio Test with $a_n = \frac{n!}{e^n}$, we have $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\frac{(n+1)!}{e^{n+1}} \cdot \frac{e^n}{n!} \right] = \lim_{n\to\infty} \frac{n+1}{e} = \infty$, so the series diverges.

15. $\sum_{n=1}^{\infty} \frac{2^n}{n!n}$. We use the Ratio Test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left[\frac{2^{(n+1)}}{(n+1)!\,(n+1)} \cdot \frac{n!n}{2^n} \right] = \lim_{n\to\infty} \frac{2n}{(n+1)^2} = 0$, so the series converges.

34. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(3n)!}$. We use the Ratio Test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left\{ \frac{[(n+1)!]^2}{(3n+3)!} \cdot \frac{(3n)!}{(n!)^2} \right\} = \lim_{n\to\infty} \frac{(n+1)^2}{(3n+1)\,(3n+2)\,(3n+3)} = 0$.

Thus, the series converges absolutely.