9.4 The Comparison Tests

5. $\dfrac{1}{\sqrt{n^2-1}} > \dfrac{1}{\sqrt{n^2}} = \dfrac{1}{n}$ for $n \geq 2$. Since $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$.

15. If n is large, then $a_n = \dfrac{n}{\sqrt{n^5-1}}$ behaves like $\dfrac{n}{\sqrt{n^5}} = \dfrac{1}{n^{3/2}} = b_n$.

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{\sqrt{n^5-1}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{5/2}}{\sqrt{n^5-1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-(1/n^5)}} = 1 > 0$, so $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ converges $\Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^5-1}}$ converges.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{(n+2)(2n^2+1)}$. If n is large, then $a_n = \dfrac{n+1}{(n+2)(2n^2+1)}$ behaves like $\dfrac{n}{n(2n^2)} = \dfrac{1}{2n^2}$, so we take $b_n = \dfrac{1}{n^2}$.

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{n+1}{(n+2)(2n^2+1)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+n^2}{(n+2)(2n^2+1)} = \frac{1}{2} > 0$, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{n+1}{(n+2)(2n^2+1)}$ converges.

38. $a_n = \dfrac{1}{1+2+3+\cdots+n} = \dfrac{1}{\frac{n(n+1)}{2}} = \dfrac{2}{n(n+1)} < \dfrac{2}{n^2}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+2+3+\cdots+n}$.