Week 4: 7.4: 13, 23, 31, 33

7.6: 17, 19, 25, 31

13. $I = \int \frac{(x-1)\,dx}{x^2 - x - 2}$. Now $\frac{x-1}{x^2-x-2} = \frac{x-1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{(A+B)x + A - 2B}{(x-2)(x+1)}$
$\Rightarrow A + B = 1$ and $A - 2B = -1 \Rightarrow B = \frac{2}{3}$ and $A = \frac{1}{3}$, so
$I = \frac{1}{3}\int \frac{dx}{x-2} + \frac{2}{3}\int \frac{dx}{x+1} = \frac{1}{3}\ln|x-2| + \frac{2}{3}\ln|x+1| + C = \frac{1}{3}\ln\left|(x-2)(x+1)^2\right| + C.$

23. $I = \int \frac{v^3 + 1}{v(v-1)^3}\,dv$. Now

$$\frac{v^3+1}{v(v-1)^3} = \frac{A}{v} + \frac{B}{v-1} + \frac{C}{(v-1)^2} + \frac{D}{(v-1)^3} = \frac{A(v-1)^3 + Bv(v-1)^2 + Cv(v-1) + Dv}{v(v-1)^3}$$

$$= \frac{A\left(v^3 - 3v^2 + 3v - 1\right) + B\left(v^3 - 2v^2 + v\right) + C\left(v^2 - v\right) + Dv}{v(v-1)^3}$$

$$= \frac{(A+B)v^3 - (3A + 2B - C)v^2 + (3A + B - C + D)v - A}{v(v-1)^3}$$

$\Rightarrow A + B = 1$, $3A + 2B - C = 0$, $3A + B - C + D = 0$, and $-A = 1$. We find $A = -1$, $B = 2$, $C = 3A + 2B = -3 + 4 = 1$, and $D = -3A - B + C = 3 - 2 + 1 = 2$. Therefore,

$$I = \int\left[\frac{-1}{v} + \frac{2}{v-1} + \frac{1}{(v-1)^2} + \frac{2}{(v-1)^3}\right]dv = -\ln|v| + 2\ln|v-1| - \frac{1}{v-1} - \frac{1}{(v-1)^2} + C$$

$$= \ln\left|\frac{(v-1)^2}{v}\right| - \frac{v}{(v-1)^2} + C$$

31. $I = \int \frac{x^3+3}{(x+1)(x^2+1)}\,dx = \int\left[1 - \frac{x^2+x-2}{(x+1)(x^2+1)}\right]dx = x - J,$

$$\begin{array}{r|l} & \quad 1 \\ x^3 + x^2 + x + 1 & x^3 \qquad\quad + 3 \\ & x^3 + x^2 + x + 1 \\ \hline & \quad -x^2 - x + 2 \end{array}$$

where $J = \int \frac{x^2+x-2}{(x+1)(x^2+1)}\,dx$. Now

$$\frac{x^2+x-2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{(A+B)x^2 + (B+C)x + (A+C)}{(x+1)(x^2+1)}$$

so $A + B = 1$, $B + C = 1$, and $A + C = -2$. Solving, we obtain $A = -1$, $B = 2$, and $C = -1$,
so $J = -\int \frac{dx}{x+1} + 2\int \frac{x\,dx}{x^2+1} - \int \frac{dx}{x^2+1} = -\ln|x+1| + \ln\left(x^2+1\right) - \tan^{-1}x + C_1$. Finally,
$I = x + \tan^{-1}x + \ln\left|\frac{x+1}{x^2+1}\right| + C.$

33. $I = \int \frac{5x^3 - 3x^2 + 7x - 3}{\left(x^2+1\right)^2}\,dx$. Now

$\frac{5x^3 - 3x^2 + 7x - 3}{\left(x^2+1\right)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{\left(x^2+1\right)^2} = \frac{Ax^3 + Bx^2 + (A+C)x + B + D}{\left(x^2+1\right)^2}$, so $A = 5$,
$B = -3$, $A + C = 7$, and $B + D = -3$. Solving, we obtain $A = 5$, $B = -3$, $C = 2$, and $D = 0$, so

$$I = \int \frac{5x\,dx}{x^2+1} - \int \frac{3\,dx}{x^2+1} + \int \frac{2x\,dx}{\left(x^2+1\right)^2} = \frac{5}{2}\ln\left(x^2+1\right) - 3\tan^{-1}x - \frac{1}{x^2+1} + C.$$

17. $\displaystyle\int_0^{\infty} \frac{x\,dx}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{x\,dx}{1+x^2} = \lim_{b\to\infty} \left[\tfrac{1}{2}\ln\left(1+x^2\right)\right]_0^b = \lim_{b\to\infty} \left[\tfrac{1}{2}\ln\left(1+b^2\right)\right] = \infty$, so the integral diverges.

19. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+4} = \int_{-\infty}^{0} \frac{dx}{x^2+4} + \int_0^{\infty} \frac{dx}{x^2+4} = 2\int_0^{\infty} \frac{dx}{x^2+4} = 2\lim_{b\to\infty} \int_0^b \frac{dx}{x^2+4} = 2\cdot\tfrac{1}{2}\lim_{b\to\infty} \tan^{-1}\left(\tfrac{1}{2}x\right)\Big|_0^b$

$\displaystyle= \lim_{b\to\infty}\left[\tan^{-1}\left(\tfrac{1}{2}b\right) - 0\right] = \frac{\pi}{2}$

25. $\displaystyle\int_0^1 \frac{dx}{x^{2/3}} = \lim_{a\to 0^+} \int_a^1 x^{-2/3}\,dx = \lim_{a\to 0^+} \left(3x^{1/3}\right)\Big|_a^1 = \lim_{a\to 0^+} 3\left(1-a^{1/3}\right) = 3$

31. $\displaystyle\int_0^4 \frac{dx}{\sqrt{x}-1} = \int_0^1 \frac{dx}{\sqrt{x}-1} + \int_1^4 \frac{dx}{\sqrt{x}-1}$. For the first integral, put $u = \sqrt{x}$, so $\displaystyle du = \frac{dx}{2\sqrt{x}} \Rightarrow dx = 2u\,du$, $x = 0 \Rightarrow u = 0$, and $x = 1 \Rightarrow u = 1$. Then

$\displaystyle\int_0^1 \frac{dx}{\sqrt{x}-1} = \int_0^1 \frac{2u\,du}{u-1} = 2\int_0^1 \left(1+\frac{1}{u-1}\right)du = [2u]_0^1 + 2\lim_{c\to 1^-} [\ln|u-1|]_0^c = -\infty$, so the integral diverges.