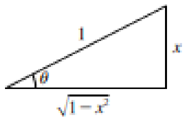
Week 3: 7.3: 9, 15, 21, 23

9. $\int x^3\sqrt{1-x^2}\,dx$. Let $x=\sin\theta$, so $dx=\cos\theta\,d\theta$ and $\sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\cos\theta$.

Then



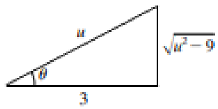
$$\begin{aligned}\int x^3\sqrt{1-x^2}\,dx &= \int \sin^3\theta\cos\theta\cos\theta\,d\theta=\int\sin^2\theta\cos^2\theta\sin\theta\,d\theta\\ &= \int\left(1-\cos^2\theta\right)\cos^2\theta\sin\theta\,d\theta=\int\left(\cos^2\theta-\cos^4\theta\right)\sin\theta\,d\theta\\ &= -\tfrac{1}{3}\cos^3\theta+\tfrac{1}{5}\cos^5\theta+C=-\tfrac{1}{15}\left(\cos^3\theta\right)\left(5-3\cos^2\theta\right)+C\\ &= -\tfrac{1}{15}\left(1-x^2\right)^{3/2}\left[5-3\left(1-x^2\right)\right]+C=-\tfrac{1}{15}\left(3x^2+2\right)\left(1-x^2\right)^{3/2}+C\end{aligned}$$

Note that the problem can also be solved using the substitution $u=1-x^2$.

15. $\displaystyle\int\frac{\sqrt{16x^2-9}}{x}\,dx$. Let $u=4x$, so $du=4\,dx$. Then $\displaystyle\int\frac{\sqrt{16x^2-9}}{x}\,dx=\int\frac{\sqrt{u^2-9}}{u}\,du$.

Next, let $u=3\sec\theta$, so $du=3\sec\theta\tan\theta\,d\theta$ and

$\sqrt{u^2-9}=\sqrt{9\sec^2\theta-9}=3\sqrt{\sec^2\theta-1}=3\tan\theta$. Then



$$\begin{aligned}\int\frac{\sqrt{u^2-9}}{u}\,du &= \int\frac{(3\tan\theta)(3\sec\theta\tan\theta)}{3\sec\theta}\,d\theta=3\int\tan^2\theta\,d\theta=3\int\left(\sec^2\theta-1\right)d\theta=3(\tan\theta-\theta)+C\\ &= 3\left[\tfrac{1}{3}\sqrt{u^2-9}-\sec\left(\tfrac{1}{3}u\right)\right]+C=\sqrt{16x^2-9}-3\sec^{-1}\left(\tfrac{4}{3}x\right)+C\end{aligned}$$

21. $I=\int_{-\sqrt{3}}^{\sqrt{3}}\sqrt{4-x^2}\,dx=2\int_0^{\sqrt{3}}\sqrt{4-x^2}\,dx$. Let $x=2\sin\theta$, so $dx=2\cos\theta\,d\theta$, $\sqrt{4-x^2}=\sqrt{4-4\sin^2\theta}=2\cos\theta$,

$x=0\Rightarrow\theta=0$, and $x=\sqrt{3}\Rightarrow\theta=\frac{\pi}{3}$. Then

$$\begin{aligned}I &= 2\int_0^{\pi/3}(2\cos\theta)(2\cos\theta)\,d\theta=8\int_0^{\pi/3}\cos^2\theta\,d\theta=4\int_0^{\pi/3}(1+\cos2\theta)\,d\theta=4\left(\theta+\tfrac{1}{2}\sin2\theta\right)\Big|_0^{\pi/3}\\ &= 4\left(\tfrac{\pi}{3}+\tfrac{\sqrt{3}}{4}\right)=\tfrac{1}{3}\left(4\pi+3\sqrt{3}\right)\end{aligned}$$

23. $I=\displaystyle\int_1^{\sqrt{3}}\frac{dx}{\left(1+x^2\right)^{3/2}}$. Let $x=\tan\theta$, so $dx=\sec^2\theta\,d\theta$, $\sqrt{1+x^2}=\sqrt{1+\tan^2\theta}=\sec\theta$, $x=1\Rightarrow\theta=\frac{\pi}{4}$, and

$x=\sqrt{3}\Rightarrow\theta=\frac{\pi}{3}$. Then $I=\displaystyle\int_{\pi/4}^{\pi/3}\frac{\sec^2\theta}{\sec^3\theta}\,d\theta=\int_{\pi/4}^{\pi/3}\cos\theta\,d\theta=\sin\theta\Big|_{\pi/4}^{\pi/3}=\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}=\frac{1}{2}\left(\sqrt{3}-\sqrt{2}\right)$.