Calculus Exam-1 (106.10.24)

- **1. (15%) Find**

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$$

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- Sol :

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)}=\lim_{x\to 0}\frac{1}{\sqrt{1+x}+1}=\frac{1}{2}$$

- **2. (15%) Let**

$$f(x)=\begin{cases}\frac{x^2-4}{x+2} & \textbf{if } x\neq -2\\ k & \textbf{if } x=-2\end{cases}$$

Find the value of k that will make f continuous on $(-\infty,\infty)$..

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- Sol : Let f is continuous on -2 then we have

$$f(-2)=k=\lim_{x\to -2}\left(\frac{x^2-4}{x+2}\right)=\lim_{x\to -2}(x-2)=-4 \;\Rightarrow\; \therefore\; k=-4$$

- **3. (15%) Use the definition of the derivative to find the derivative of $y=1/(x+1)$**

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- Sol :

$$\begin{aligned}\frac{d}{dx}y(x) &= \lim_{h\to 0}\frac{y(x+h)-y(x)}{h}=\lim_{h\to 0}\left(\frac{\frac{1}{x+h+1}-\frac{1}{x+1}}{h}\right)\\ &= \lim_{h\to 0}\frac{x+1-(x+h+1)}{h(x+h+1)(x+1)}=\lim_{h\to 0}\frac{-1}{(x+h+1)(x+1)}=\frac{-1}{(x+1)^2}\end{aligned}$$

- **4. (15%) How fast is $y = (\frac{2t-1}{t^2+1})^5$ changing when $t = 1$**

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∘ Sol :

$$\text{changing rate} = \frac{d}{dt}\, y(t) = 5\left(\frac{2t-1}{t^2+1}\right)^4 \cdot \left(\frac{2(t^2+1)-(2t-1)2t}{(t^2+1)^2}\right) = 5\left(\frac{2t-1}{t^2+1}\right)^4 \cdot \left(\frac{-2t^2+2t+2}{(t^2+1)^2}\right)$$

Hence when $t = 1$ we have

$$\left.\frac{d}{dt}\, y(t)\right|_{t=1} = 5 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{2}{4}\right) = 5 \cdot \left(\frac{1}{2}\right)^5 = \frac{5}{32}$$

Therefore y is increasing at the rate of $5/32$ units per unit change in t when $t = 1$.

- **5. (10%) Show that $f(x) = x^3 + x - 1$ has at least one zero in $(0, 1)$.**

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▷ Proof :

Since f is a continuous function on $[0, 1]$ and

$$f(0) = -1 < 0 \quad ; \quad f(1) = 1 > 0$$

then according to the Intermediate Value Theorem we know that there is at least one root of the equation $f(x) = 0$ for $x \in (0, 1)$

- **6. (10%) Use the Product Rule to find the derivative of $f(x) = (2 + 3x^2)(x^3 - 5)$**

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▷ Sol :

$$f'(x) = 6x(x^3 - 5) + (2 + 3x^2)3x^2 = 6x^4 - 30x + 6x^2 + 9x^4 = 15x^4 + 6x^2 - 30x$$

- **7. (10%) Find the derivative of**

$$y = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$$

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▷ Sol :

$$y'(x) = \frac{(\sqrt{x}-1)'(\sqrt{x}+1) - (\sqrt{x}-1)(\sqrt{x}+1)'}{(\sqrt{x}+1)^2} = \frac{(\sqrt{x}+1) - (\sqrt{x}-1)}{2\sqrt{x}(\sqrt{x}+1)^2} = \frac{1}{\sqrt{x}(\sqrt{x}+1)^2} \quad \text{or} \quad \frac{1}{x^{3/2} + 2x + x^{1/2}}$$

- **8. (10%) Find the derivative of**

$$\boldsymbol{y = \frac{\cos x}{1+x}}$$

▷ Sol :

$$y'(x) = \frac{-\sin x(1+x) - \cos x \cdot 1}{(1+x)^2} = \frac{-\sin x - x\sin x - \cos x}{(1+x)^2}$$