1.4 Continuous Functions

27. We require that $f(1) = 1 + 2 = 3 = \lim_{x \to 1^+} kx^2 = k$, or $k = 3$.

28. Since $\lim_{x \to -2} \frac{x^2 - 4}{x + 2} = \lim_{x \to -2} \frac{(x-2)(x+2)}{x+2} = \lim_{x \to -2} (x - 2) = -4$, we define $f(-2) = k = -4$, that is, take $k = -4$.

49. The absolute value function is continuous, so $\lim_{x \to 2} \left| \frac{x^2 + x - 6}{x - 2} \right| = \left| \lim_{x \to 2} \frac{(x+3)(x-2)}{x-2} \right| = \left| \lim_{x \to 2} (x + 3) \right| = |5| = 5$.

51. $\lim_{x \to 0} \frac{3x^3 - 2x}{5x} = \lim_{x \to 0} \frac{x\left(3x^2 - 2\right)}{5x} = \lim_{x \to 0} \frac{3x^2 - 2}{5} = -\frac{2}{5}$. So f will be continuous at 0 if we define

$$f(x) = \begin{cases} \frac{3x^3 - 2x}{5x} & \text{if } x \neq 0 \\ -\frac{2}{5} & \text{if } x = 0 \end{cases}$$

59. $f(x) = x^2 - x + 1$ is continuous on $[-1, 4]$. $f(-1) = 3$ and $f(4) = 13$. Since $f(-1) < 7 < f(4)$, there exists at least one c in $[-1, 4]$ such that $f(c) = 7$. To find c we solve $x^2 - x + 1 = 7 \Rightarrow x^2 - x - 6 = (x - 3)(x + 2) = 0 \Rightarrow x = -2$ or 3. Since -2 does not lie in $[-1, 4]$, we see that $c = 3$.

1.5 Tangent Lines and Rates of Change

15. $$\lim_{h \to 0} \frac{f(1+h) - f(1)}{(1+h) - 1} = \lim_{h \to 0} \frac{\left[2(1+h)^2 + 1\right] - \left[2(1)^2 + 1\right]}{h} = \lim_{h \to 0} \frac{2 + 4h + 2h^2 + 1 - 3}{h}$$
$$= \lim_{h \to 0} \frac{h(4 + 2h)}{h} = 4$$

19. $$\lim_{h \to 0} \frac{f(1+h) - f(1)}{(1+h) - 1} = \lim_{h \to 0} \frac{\left[\frac{2}{1+h} + (1+h)\right] - \left(\frac{2}{1} + 1\right)}{h} = \lim_{h \to 0} \frac{\frac{2}{1+h} + h - 2}{h}$$
$$= \lim_{h \to 0} \frac{2 + h + h^2 - 2 - 2h}{h(1+h)} = \lim_{h \to 0} \frac{h(h-1)}{h(1+h)} = -1$$

28. $V(r) = \frac{4}{3}\pi r^3$ gives the volume of a sphere.

a. $V(1) = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$ and $V(2) = \frac{4}{3}\pi(2)^3 = \frac{32}{3}\pi$, so the average rate of change over $[1, 2]$ is $\frac{V(2) - V(1)}{2 - 1} = \frac{\frac{32}{3}\pi - \frac{4}{3}\pi}{1} = \frac{28}{3}\pi$ units^3/unit.

b. $$\lim_{h \to 0} \frac{V(2+h) - V(2)}{(2+h) - 2} = \lim_{h \to 0} \frac{\frac{4}{3}\pi(2+h)^3 - \frac{4}{3}\pi(2)^3}{h} = \frac{4}{3}\pi \lim_{h \to 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h}$$
$$= \frac{4}{3}\pi \lim_{h \to 0} \frac{h\left(12 + 6h + h^2\right)}{h} = 16\pi \text{ units}^3/\text{unit}$$