13.7 Tangent Planes and Normal Lines ET 12.7

6. $F(x,y) = x^4 - x^2 + y^2 \Rightarrow \nabla F\left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right) = \left[\left(4x^3 - 2x\right)\mathbf{i} + 2y\mathbf{j}\right]_{(1/2,\sqrt{3}/4)} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$ is normal to the level curve $F(x,y) = x^4 - x^2 + y^2 = 0$ at $\left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right)$. So the slope of the required normal line is $m = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$ and an equation of the normal line is $y - \frac{\sqrt{3}}{4} = -\sqrt{3}\left(x - \frac{1}{2}\right) \Leftrightarrow y = -\sqrt{3}x + \frac{3\sqrt{3}}{4}$. The slope of the required tangent line is $m = -\frac{1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$, and so an equation of the tangent line is $y - \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{3}\left(x - \frac{1}{2}\right) \Leftrightarrow y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{12}$.

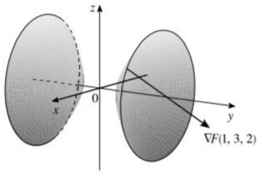
13. $F(1,3,2) = \left(-x^2 + y^2 - z^2\right)\Big|_{(1,3,2)} = -1+9-4 = 4$,

so an equation of the required level surface is

$-x^2 + y^2 - z^2 = 4 \Leftrightarrow \frac{y^2}{2^2} - \frac{x^2}{2^2} - \frac{z^2}{2^2} = 1$.

$\nabla F(1,3,2) = (-2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k})|_{(1,3,2)}$

$= -2\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$.



20. $F(x,y,z) = xyz + 4 = 0 \Rightarrow \nabla F(2,-1,2) = (yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k})|_{(2,-1,2)} = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} = -2(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$, so an equation of the tangent plane at $(2,-1,2)$ is $(x-2) - 2(y+1) + (z-2) = 0 \Leftrightarrow x - 2y + z = 6$. Equations of the normal line passing through $(2,-1,2)$ are $\frac{x-2}{1} = \frac{y+1}{-2} = \frac{z-2}{1} \Leftrightarrow x - 2 = \frac{y+1}{-2} = z - 2$.

33. $F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \Rightarrow \nabla F(x_0,y_0,z_0) = \left(\frac{2x}{a^2}\mathbf{i} + \frac{2y}{b^2}\mathbf{j} + \frac{2z}{c^2}\mathbf{k}\right)\Big|_{(x_0,y_0,z_0)} = \frac{2x_0}{a^2}\mathbf{i} + \frac{2y_0}{b^2}\mathbf{j} + \frac{2z_0}{c^2}\mathbf{k}$, so an equation of the tangent plane at (x_0,y_0,z_0) is $\frac{2x_0}{a^2}(x - x_0) + \frac{2y_0}{b^2}(y - y_0) + \frac{2z_0}{c^2}(z - z_0) = 0 \Leftrightarrow$ $\frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{z_0z}{c^2} - \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}\right) = 0$. But (x_0,y_0,z_0) lies on the ellipsoid, so the expression in parentheses is equal to 1 and we have $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1$, as was to be shown.

13.8 Extrema of Functions of Two Variables ET 12.8

8. $\left.\begin{array}{l} f_x(x,y) = \frac{\partial}{\partial x}\left(x^2 + 3y^2 - 6xy - 2x + 4y\right) = 2x - 6y - 2 = 0 \\ f_y(x,y) = \frac{\partial}{\partial y}\left(x^2 + 3y^2 - 6xy - 2x + 4y\right) = 6y - 6x + 4 = 0 \end{array}\right\} \Rightarrow x = \frac{1}{2}, y = -\frac{1}{6}$, so $\left(\frac{1}{2}, -\frac{1}{6}\right)$ is the sole critical point of f. Next, $D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - f_{xy}^2(x,y) = 2 \cdot 6 - (-6)^2 = -24$. Since $D\left(\frac{1}{2}, -\frac{1}{6}\right) = -24 < 0$, we conclude that $\left(\frac{1}{2}, -\frac{1}{6}, -\frac{5}{6}\right)$ is a saddle point of f.

13. $\left.\begin{array}{l} f_x(x,y) = \frac{\partial}{\partial x}\left(x^2 - 6x - x\sqrt{y} + y\right) = 2x - 6 - \sqrt{y} = 0 \\ f_y(x,y) = \frac{\partial}{\partial y}\left(x^2 - 6x - xy^{1/2} + y\right) = -\frac{x}{2\sqrt{y}} + 1 = 0 \end{array}\right\}$ From the first equation, we see that $\sqrt{y} = 2x - 6$. Substituting this into the second equation gives $-x + 2(2x - 6) = 0 \Rightarrow x = 4$. Substituting this into the first equation gives $8 - 6 = \sqrt{y} \Rightarrow y = 4$, so the sole critical point of f is $(4,4)$. Next,

$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - f_{xy}^2(x,y) = 2\left(\frac{x}{4y^{3/2}}\right) - \left(-\frac{1}{2\sqrt{y}}\right)^2 = \frac{x}{2y^{3/2}} - \frac{1}{4y}$. Since $D(4,4) = \frac{4}{2(8)} - \frac{1}{4(4)} = \frac{3}{16} > 0$ and $f_{xx}(4,4) = 2 > 0$, the point $(4,4)$ gives a relative minimum of f with value $f(4,4) = -12$.

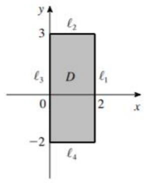
15. $\left.\begin{array}{l} f_x(x,y) = \dfrac{\partial}{\partial x}\left(xy - \dfrac{2}{x} - \dfrac{4}{y}\right) = y + \dfrac{2}{x^2} = \dfrac{x^2y+2}{x^2} = 0 \\ f_y(x,y) = \dfrac{\partial}{\partial y}\left(xy - \dfrac{2}{x} - \dfrac{4}{y}\right) = \dfrac{xy^2+4}{y^2} = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x^2y+2=0 \\ xy^2+4=0 \end{array}\right\}$ From the first equation,

$y = -\dfrac{2}{x^2}$, and substituting this into the second equation yields $x\left(-\dfrac{2}{x^2}\right)^2 + 4 = 0 \Leftrightarrow 4\left(1+x^3\right) = 0 \Rightarrow x = -1$.

Substituting into the first equation gives $y = -2$, so $(-1,-2)$ is the only critical point of f. Next, $f_{xx}(x,y) = -\dfrac{4}{x^3}$,

$f_{xy}(x,y) = 1$, and $f_{yy}(x,y) = -\dfrac{8}{y^3}$, so $D(x,y) = f_{xx}(x,y)\,f_{yy}(x,y) - f_{xy}^2(x,y) = \dfrac{32}{x^3y^3} - 1$. Since $D(-1,-2) = 3 > 0$ and $f_{xx}(-1,-2) = 4 > 0$, the point $(-1,-2)$ gives a relative minimum of f with value $f(-1,-2) = (-1)(-2) - \dfrac{2}{-1} - \dfrac{4}{-2} = 6$.

33. Since $f_x(x,y) = \frac{\partial}{\partial x}(2x+3y-6) = 2$ and $f_y(x,y) = \frac{\partial}{\partial y}(2x+3y-6) = 3$ are never equal to 0, f has no critical point on D.



On ℓ_1, $x = 2$ and $y = y$, so $g(y) = f(2,y) = 3y - 2$ for $-2 \le y \le 3$. We see that g has an absolute minimum value of -8 at $(2,-2)$ and an absolute maximum value of 7 at $(2,3)$.

On ℓ_2, $x = x$ and $y = 3$, so $h(x) = f(x,3) = 2x + 3$ for $0 \le x \le 2$. We see that h has an absolute minimum value of 3 at $(0,3)$ and an absolute maximum value of 7 at $(2,3)$.

On ℓ_3, $x = 0$ and $y = y$, so $s(y) = f(0,y) = 3y - 6$ for $-2 \le y \le 3$. We see that s has an absolute minimum value of -12 at $(0,-2)$ and an absolute maximum value of 3 at $(0,3)$.

On ℓ_4, $x = x$ and $y = -2$, so $t(x) = f(x,-2) = 2x - 12$ for $0 \le x \le 2$. We see that t has an absolute minimum value of -12 at $(0,-2)$ and an absolute maximum value of -8 at $(2,-2)$.

The extreme values of f on each boundary of D are summarized below.

	ℓ_1		ℓ_2		ℓ_3		ℓ_4	
(x, y)	$(2,-2)$	$(2,3)$	$(0,3)$	$(2,3)$	$(0,-2)$	$(0,3)$	$(0,-2)$	$(2,-2)$
Extreme value	-8	7	3	7	-12	3	-12	-8

We see that f has an absolute minimum value of $f(0,-2) = -12$ and an absolute maximum value of $f(2,3) = 7$ on D.