7 Techniques of Integration

7.1 Integration by Parts

7. $I = \int x^2 e^{-x}\,dx$. Let $u = x^2$ and $dv = e^{-x}\,dx$. Then $du = 2x\,dx$ and $v = \int e^{-x}\,dx = -e^{-x}$, so $I = uv - \int v\,du = -x^2 e^{-x} + \int 2xe^{-x}\,dx$. To evaluate $J = \int xe^{-x}\,dx$, let $s = x$ and $dt = e^{-x}\,dx$. Then $ds = dx$ and $t = \int e^{-x}\,dx = -e^{-x}$, so $J = st - \int t\,ds = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x}$, and therefore $I = \int x^2 e^{-x}\,dx = -x^2 e^{-x} + 2J = -x^2 e^{-x} + 2\left(-xe^{-x} - e^{-x}\right) + C = -\left(x^2 + 2x + 2\right)e^{-x} + C$.

11. $\int \tan^{-1} x\,dx$. Let $u = \tan^{-1} x$ and $dv = dx$. Then $du = \dfrac{dx}{1+x^2}$ and $v = x$, so $\int \tan^{-1} x\,dx = uv - \int v\,du = x\tan^{-1} x - \int \dfrac{x\,dx}{1+x^2} = x\tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right) + C$.

34. $\int_0^2 \ln(x+1)\,dx$. Let $u - x + 1$, so $du - dx$, $x - 0 \to u - 1$, and $x - 2 \to u - 3$. Then $\int_0^2 \ln(x+1)\,dx = \int_1^3 \ln u\,du = (u\ln u - u)|_1^3 = 3\ln 3 - 2$. (See Example 6.)

35. $\int_0^{1/2} \cos^{-1} x\,dx$. Let $u = \cos^{-1} x$ and $dv = dx$, so $du = -dx \Big/ \sqrt{1-x^2}$ and $v = x$. Then

$$\int_0^{1/2} \cos^{-1} x\,dx = x\cos^{-1} x\Big|_0^{1/2} + \int x\left(1-x^2\right)^{-1/2} dx = \tfrac{1}{2}\cdot\tfrac{\pi}{3} + \left[\left(-\tfrac{1}{2}\right)2\left(1-x^2\right)^{1/2}\right]_0^{1/2} = \tfrac{\pi}{6} - \tfrac{\sqrt{3}}{2} + 1$$
$$= \tfrac{1}{6}\left(\pi + 6 - 3\sqrt{3}\right)$$

7.2 Trigonometric Integrals

7. $\displaystyle\int_0^{\pi} \cos^2\frac{x}{2}\,dx = \int_0^{\pi} \frac{1+\cos x}{2}\,dx = \frac{x+\sin x}{2}\Big|_0^{\pi} = \frac{\pi}{2}$

19. $\int_0^{\pi/4} \tan^2 x\,dx = \int_0^{\pi/4}\left(\sec^2 x - 1\right)dx = (\tan x - x)|_0^{\pi/4} = 1 - \frac{\pi}{4} = \frac{4-\pi}{4}$

39. $\int_0^{\pi/2} \sin x\cos 2x\,dx = \frac{1}{2}\int_0^{\pi/2}[\sin(1-2)x + \sin(1+2)x]\,dx = \frac{1}{2}\int_0^{\pi/2}(-\sin x + \sin 3x)\,dx$

$$= \tfrac{1}{2}\left(\cos x - \tfrac{1}{3}\cos 3x\right)\Big|_0^{\pi/2} = \tfrac{1}{2}\left(0 - \tfrac{2}{3}\right) = -\tfrac{1}{3}$$

47. $\displaystyle\int \frac{1-\tan^2 x}{\sec^2 x}\,dx = \int \frac{1-\left(\sec^2 x - 1\right)}{\sec^2 x}\,dx = \int \frac{2-\sec^2 x}{\sec^2 x}\,dx = \int\left(2\cos^2 x - 1\right)dx = \int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$