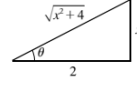


7.3 Trigonometric Substitutions

7. $\int \frac{dx}{x^2\sqrt{x^2+4}}$. Let $x = 2 \tan \theta$, so $dx = 2 \sec^2 \theta d\theta$ and

$$\sqrt{x^2+4} = \sqrt{4 \tan^2 \theta + 4} = 2\sqrt{\tan^2 \theta + 1} = 2 \sec \theta. \text{ Then}$$

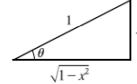
$$\begin{aligned} \int \frac{dx}{x^2\sqrt{x^2+4}} &= \int \frac{2 \sec^2 \theta d\theta}{(4 \tan^2 \theta)(2 \sec \theta)} = \frac{1}{4} \int \frac{\sec \theta}{\tan^2 \theta} d\theta = \frac{1}{4} \int (\sin \theta)^{-2} \cos \theta d\theta \\ &= -\frac{1}{4 \sin \theta} + C = -\frac{\sqrt{x^2+4}}{4x} + C \end{aligned}$$



9. $\int x^3 \sqrt{1-x^2} dx$. Let $x = \sin \theta$, so $dx = \cos \theta d\theta$ and $\sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \cos \theta$.

Then

$$\begin{aligned} \int x^3 \sqrt{1-x^2} dx &= \int \sin^3 \theta \cos \theta \cos \theta d\theta = \int \sin^2 \theta \cos^2 \theta \sin \theta d\theta \\ &= \int (1 - \cos^2 \theta) \cos^2 \theta \sin \theta d\theta = \int (\cos^2 \theta - \cos^4 \theta) \sin \theta d\theta \\ &= -\frac{1}{3} \cos^3 \theta + \frac{1}{5} \cos^5 \theta + C = -\frac{1}{15} (\cos^3 \theta) (5 - 3 \cos^2 \theta) + C \\ &= -\frac{1}{15} (1-x^2)^{3/2} [5 - 3(1-x^2)] + C = -\frac{1}{15} (3x^2 + 2) (1-x^2)^{3/2} + C \end{aligned}$$

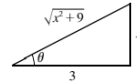


Note that the problem can also be solved using the substitution $u = 1 - x^2$.

11. $\int \frac{x^3 dx}{\sqrt{x^2+9}}$. Let $x = 3 \tan \theta$, so $dx = 3 \sec^2 \theta d\theta$ and

$$\sqrt{x^2+9} = \sqrt{9 \tan^2 \theta + 9} = 3\sqrt{\tan^2 \theta + 1} = 3 \sec^2 \theta. \text{ Then}$$

$$\begin{aligned} \int \frac{x^3 dx}{\sqrt{x^2+9}} &= \int \frac{(3 \tan \theta)^3 (3 \sec^2 \theta d\theta)}{3 \sec \theta} = 27 \int \tan^3 \theta \sec \theta d\theta \\ &= 27 \int \tan^2 \theta \sec \theta \tan \theta d\theta = 27 \int (\sec^2 \theta - 1) \sec \theta \tan \theta d\theta = 27 \left(\frac{1}{3} \sec^3 \theta - \sec \theta \right) + C \\ &= 9 \sec \theta (\sec^2 \theta - 3) + C = 9 \frac{\sqrt{x^2+9}}{3} \left(\frac{x^2+9}{9} - 3 \right) + C = \frac{1}{3} (x^2 - 18) \sqrt{x^2+9} + C \end{aligned}$$



Note that the problem can also be solved using the substitution $u = x^2 + 9$.

12. $I = \int_0^{3/4} \frac{x^2 dx}{\sqrt{9-4x^2}}$. Let $u = 2x$, so $du = 2 dx$, $x = 0 \Rightarrow u = 0$, and $x = \frac{3}{4} \Rightarrow u = \frac{3}{2}$. Then

$$\int_0^{3/4} \frac{x^2 dx}{\sqrt{9-4x^2}} = \int_0^{3/2} \frac{\left(\frac{1}{2}u\right)^2 \left(\frac{1}{2} du\right)}{\sqrt{9-u^2}} = \frac{1}{8} \int_0^{3/2} \frac{u^2 du}{\sqrt{9-u^2}}. \text{ Now let } u = 3 \sin \theta, \text{ so } du = 3 \cos \theta d\theta. \text{ Then}$$

$$\sqrt{9-u^2} = \sqrt{9-9 \sin^2 \theta} = 3\sqrt{1-\sin^2 \theta} = 3 \cos \theta, u = 0 \Rightarrow \theta = 0, \text{ and } u = \frac{3}{2} \Rightarrow 3 \sin \theta = \frac{3}{2} \Rightarrow \theta = \frac{\pi}{6}.$$

Thus,

$$\begin{aligned} I &= \frac{1}{8} \int_0^{3/2} \frac{u^2 du}{\sqrt{9-u^2}} = \frac{1}{8} \int_0^{\pi/6} \frac{(3 \sin \theta)^2 (3 \cos \theta d\theta)}{3 \cos \theta} = \frac{9}{8} \int_0^{\pi/6} \sin^2 \theta d\theta = \frac{9}{8} \int_0^{\pi/6} \frac{1 - \cos 2\theta}{2} d\theta \\ &= \frac{9}{16} \left(\theta - \frac{1}{2} \sin 2\theta \right) \Big|_0^{\pi/6} = \frac{9}{16} \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} \right) = \frac{3}{64} (2\pi - 3\sqrt{3}) \end{aligned}$$

7.4 The Method of Partial Fractions

7. $I = \int \frac{dx}{x(x-4)}$. Now $\frac{1}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4} = \frac{(A+B)x - 4A}{x(x-4)} \Rightarrow A+B=0$ and $-4A=1 \Rightarrow A = -\frac{1}{4}$ and

$$B = \frac{1}{4}, \text{ so } I = -\frac{1}{4} \int \frac{dx}{x} + \frac{1}{4} \int \frac{dx}{x-4} = -\frac{1}{4} \ln|x| + \frac{1}{4} \ln|x-4| + C = \frac{1}{4} \ln \left| \frac{x-4}{x} \right| + C.$$

21. $I = \int \frac{4x^2+3x+2}{x^3+x^2} dx$. Now

$$\begin{aligned} \frac{4x^2+3x+2}{x^3+x^2} &= \frac{4x^2+3x+2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} = \frac{Ax(x+1) + B(x+1) + Cx^2}{x^2(x+1)} \\ &= \frac{A(x^2+x) + B(x+1) + Cx^2}{x^2(x+1)} = \frac{(A+C)x^2 + (A+B)x + B}{x^2(x+1)} \end{aligned}$$

$\Rightarrow A+C = 4, A+B = 3, \text{ and } B = 2, \text{ leading to } A = 1 \text{ and } C = 3. \text{ Thus,}$

$$I = \int \left(\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x+1} \right) dx = \ln|x| - \frac{2}{x} + 3 \ln|x+1| + C = \ln|x(x+1)^3| - \frac{2}{x} + C.$$

$$31. I = \int \frac{x^3 + 3}{(x+1)(x^2+1)} dx = \int \left[1 - \frac{x^2 + x - 2}{(x+1)(x^2+1)} \right] dx = x - J,$$

$$\text{where } J = \int \frac{x^2 + x - 2}{(x+1)(x^2+1)} dx. \text{ Now}$$

$$\frac{x^2 + x - 2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{(A+B)x^2 + (B+C)x + (A+C)}{(x+1)(x^2+1)}$$

so $A+B=1$, $B+C=1$, and $A+C=-2$. Solving, we obtain $A=-1$, $B=2$, and $C=-1$,

$$\text{so } J = -\int \frac{dx}{x+1} + 2\int \frac{x dx}{x^2+1} - \int \frac{dx}{x^2+1} = -\ln|x+1| + \ln(x^2+1) - \tan^{-1}x + C_1. \text{ Finally,}$$

$$I = x + \tan^{-1}x + \ln \left| \frac{x+1}{x^2+1} \right| + C.$$

$$33. I = \int \frac{5x^3 - 3x^2 + 7x - 3}{(x^2+1)^2} dx. \text{ Now}$$

$$\frac{5x^3 - 3x^2 + 7x - 3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} = \frac{Ax^3 + Bx^2 + (A+C)x + B + D}{(x^2+1)^2}, \text{ so } A=5,$$

$B=-3$, $A+C=7$, and $B+D=-3$. Solving, we obtain $A=5$, $B=-3$, $C=2$, and $D=0$, so

$$I = \int \frac{5x dx}{x^2+1} - \int \frac{3 dx}{x^2+1} + \int \frac{2x dx}{(x^2+1)^2} = \frac{5}{2} \ln(x^2+1) - 3 \tan^{-1}x - \frac{1}{x^2+1} + C.$$

$$x^3 + x^2 + x + 1 \left| \frac{x^3 + 3}{x^3 + x^2 + x + 1} \right|$$

$$\frac{x^3 + x^2 + x + 1}{-x^2 - x + 2}$$