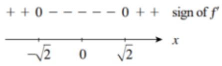
3.3 Increasing and Decreasing Functions and the First Derivative Test

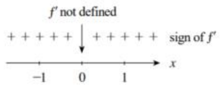
11. $f(x) = x^3 - 6x + 1 \Rightarrow f'(x) = 3x^2 - 6 = 3\left(x^2 - 2\right) = 3\left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)$ is continuous everywhere and has zeros at $-\sqrt{2}$ and $\sqrt{2}$, the critical numbers of f. The sign diagram of f' is shown.



a. f is increasing on $\left(-\infty, -\sqrt{2}\right)$ and $\left(\sqrt{2}, \infty\right)$ and decreasing on $\left(-\sqrt{2}, \sqrt{2}\right)$.

b. f has a relative maximum of $f\left(-\sqrt{2}\right) = 1 + 4\sqrt{2}$ and a relative minimum of $f\left(\sqrt{2}\right) = 1 - 4\sqrt{2}$.

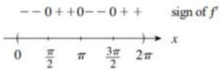
17. $f(x) = x^{1/3} - 1 \Rightarrow f'(x) = \frac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$ is discontinuous at 0 and has no zero. Therefore 0 is the only critical number of f. The sign diagram of f' is shown.



a. f is increasing on $(-\infty, \infty)$.

b. f has no relative extremum.

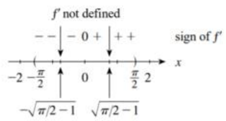
33. $f(x) = \cos^2 x$, $0 < x < 2\pi \Rightarrow$
$f'(x) = (2\cos x)(-\sin x) = -2\sin x \cos x = -\sin 2x$ is continuous on $(0, 2\pi)$ and has zeros where $\sin 2x = 0 \Rightarrow x = \frac{\pi}{2}$, π, or $\frac{3\pi}{2}$ in $(0, 2\pi)$. These are the critical numbers of f. The sign diagram of f' is shown.



a. f is decreasing on $\left(0, \frac{\pi}{2}\right)$ and $\left(\pi, \frac{3\pi}{2}\right)$ and increasing on $\left(\frac{\pi}{2}, \pi\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$.

b. f has relative minima of $f\left(\frac{\pi}{2}\right) = f\left(\frac{3\pi}{2}\right) = 0$ and a relative maximum of $f(\pi) = 1$.

37. $f(x) = \tan\left(x^2 + 1\right)$, $-\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow$
$f'(x) = \sec^2\left(x^2 + 1\right) \cdot 2x = \dfrac{2x}{\cos^2\left(x^2 + 1\right)}$ is discontinuous where
$\cos^2\left(x^2 + 1\right) = 0 \Leftrightarrow x^2 + 1 = \frac{\pi}{2} + n\pi \Leftrightarrow x = \pm\sqrt{\frac{\pi}{2} + n\pi - 1}$,
$n = 0, 1, 2, \ldots$, and has a zero at $x = 0$. Only $\pm\sqrt{\frac{\pi}{2} - 1}$ lie on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, but they are not in the domain of f, so 0 is the only critical number. The sign diagram of f' is shown.



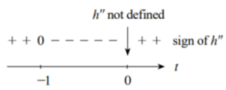
a. f is decreasing on $\left(-\frac{\pi}{2}, -\sqrt{\frac{\pi}{2} - 1}\right)$ and $\left(-\sqrt{\frac{\pi}{2} - 1}, 0\right)$ and increasing on $\left(0, \sqrt{\frac{\pi}{2} - 1}\right)$ and $\left(\sqrt{\frac{\pi}{2} - 1}, \frac{\pi}{2}\right)$.

b. f has a relative minimum of $f(0) \approx 1.56$.

3.4 Concavity and Inflection Points

17. $h(t) = \frac{1}{3}t^2 + \frac{3}{5}t^{5/3} \Rightarrow h'(t) = \frac{2}{3}t + t^{2/3} \Rightarrow h''(t) = \frac{2}{3} + \frac{2}{3}t^{-1/3} = \frac{2}{3}\left(1 + t^{-1/3}\right) = \dfrac{2\left(t^{1/3} + 1\right)}{3t^{1/3}}$.

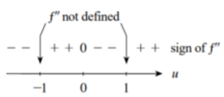
The sign diagram of h'' is shown at right. We see that h is concave upward on $(-\infty, -1)$ and $(0, \infty)$ and concave downward on $(-1, 0)$. It has inflection points at $\left(-1, -\frac{4}{15}\right)$ and $(0, 0)$.



23. $f(u) = \dfrac{u}{u^2 - 1} \Rightarrow f'(u) = \dfrac{\left(u^2 - 1\right)(1) - u(2u)}{\left(u^2 - 1\right)^2} = -\dfrac{u^2 + 1}{\left(u^2 - 1\right)^2} \Rightarrow$

$$f''(u) = \frac{-\left(u^2 - 1\right)^2(2u) + \left(u^2 + 1\right)(2)\left(u^2 - 1\right)(2u)}{\left(u^2 - 1\right)^4} = \frac{(2u)\left(u^2 - 1\right)\left(-u^2 + 1 + 2u^2 + 2\right)}{\left(u^2 - 1\right)^4} = \frac{2u\left(u^2 + 3\right)}{\left(u^2 - 1\right)^3}.$$

The sign diagram of f'' is shown at right. We see that f is concave downward on $(-\infty, -1)$ and $(0, 1)$ and concave upward on $(-1, 0)$ and $(1, \infty)$. It has an inflection point at $(0, 0)$.



37. $h(t) = \frac{1}{3}t^3 - 2t^2 - 5t - 10 \Rightarrow h'(t) = t^2 - 4t - 5 = (t - 5)(t + 1) = 0 \Rightarrow t = -1$ or 5, the critical numbers of h. $h''(t) = 2t - 4 = 2(t - 2)$. We use the SDT: $h''(-1) = -6 < 0$, so h has a relative maximum of $h(-1) = -\frac{22}{3}$; and $h''(5) = 6 > 0$, so h has a relative minimum of $h(5) = -\frac{130}{3}$.

47. $f(x) = 2\sin x + \sin 2x$, $0 < x < \pi \Rightarrow$
$f'(x) = 2\cos x + 2\cos 2x = 2\cos x + 2\left(2\cos^2 x - 1\right) = 2\left(2\cos^2 x + \cos x - 1\right) = 2(2\cos x - 1)(\cos x + 1) = 0 \Rightarrow$
$x = \frac{\pi}{3}$, the only critical number in $(0, \pi)$. $f''(x) = -2\sin x - 4\sin 2x$ and $f''\left(\frac{\pi}{3}\right) = -3\sqrt{3} < 0$, so by the SDT, f has a relative maximum of $f\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$.