3.1 Extrema of Functions

29. $f'(x) = \frac{d}{dx}\left(x^3 - 6x + 2\right) = 3x^2 - 6 = 3\left(x^2 - 2\right) = 0 \quad \Leftrightarrow \quad x = \pm\sqrt{2}$, so the critical numbers are $-\sqrt{2}$ and $\sqrt{2}$.

38. $g'(t) = \frac{d}{dt}\left(4t^{1/3} + 3t^{4/3}\right) = \frac{4}{3}t^{-2/3} + 4t^{1/3} = \frac{4}{3}t^{-2/3}(1 + 3t) = \frac{4(3t+1)}{3t^{2/3}}$. $g'(t)$ is discontinuous at $t = 0$ and has a zero at $t = -\frac{1}{3}$. Since both are in the domain of g, the critical numbers of g are $-\frac{1}{3}$ and 0.

45. $h'(x) = \frac{d}{dx}\left(x^3 + 3x^2 + 1\right) = 3x^2 + 6x = 3x(x+2) = 0 \Leftrightarrow x = -2$ or 0, both of which lie on the interval $(-3, 2)$. We calculate $h(-3) = 1$, $h(-2) = 5$, $h(0) = 1$, and $h(2) = 21$, so h has an absolute minimum value of 1 attained at $x = -3$ and $x = 0$, and an absolute maximum value of 21 attained at $x = 2$.

58. $g'(x) = \frac{d}{dx}(\cos x - \sin x) = -\sin x - \cos x$ is continuous everywhere and has zeros where $-\sin x - \cos x = 0 \Leftrightarrow \tan x = -1 \Leftrightarrow x = \frac{3\pi}{4}$ or $x = \frac{7\pi}{4}$ on the interval $(0, 2\pi)$. $g(0) = 1$, $g\left(\frac{3\pi}{4}\right) = -\sqrt{2}$, $g\left(\frac{7\pi}{4}\right) = \sqrt{2}$, and $g(2\pi) = 1$, so g has an absolute minimum value of $-\sqrt{2}$ attained at $x = \frac{3\pi}{4}$, and an absolute maximum value of $\sqrt{2}$ attained at $x = \frac{7\pi}{4}$.

3.2 The Mean Value Theorem

21. $f(x) = |x| - 1 = \begin{cases} -x - 1 & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases} \Rightarrow f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$

Thus, $f'(x) \neq 0$ for x on $(-1, 0) \cup (0, 1)$. Also, f is not differentiable at $x = 0$, so $f'(x) \neq 0$ for all x in the domain. Rolle's Theorem is not contradicted because the requirement that f be differentiable on $(-1, 1)$ is not satisfied.

22. $f(x) = 1 - x^{2/3} \Rightarrow f'(x) = -\frac{2}{3}x^{-1/3} = -\frac{2}{3x^{1/3}}$. Suppose there is a number c satisfying $-1 < c < 8$ such that $f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(8) - f(-1)}{8 - (-1)}$. Then $-\frac{2}{3c^{1/3}} = \frac{\left(1 - 8^{2/3}\right) - (1 - 1)}{9} = -\frac{1}{3} \Leftrightarrow c^{1/3} = 2 \Leftrightarrow c = 8$. This contradiction shows that no such c exists. The result does not contradict the Mean Value Theorem because f is not differentiable on the interval $(-1, 8)$.

26. Let $f(x) = x^7 + 6x^5 + 2x - 6$. Then f is continuous and differentiable on $(-\infty, \infty)$. Evaluating f at $x = -1$ and $x = 1$ gives $f(-1) = -15 < 0$ and $f(1) = 3 > 0$. By the Intermediate Value Theorem, there exists a number r_1 on the interval $(-1, 1)$ such that $f(r_1) = 0$, and this shows that f has at least one zero in $(-1, 1)$. Now suppose that the equation has another root r_2, distinct from r_1, in $(-\infty, \infty)$. Then $f(r_1) = f(r_2) = 0$ and by Rolle's Theorem, there exists a number c in the interval (r_1, r_2) [or in (r_2, r_1) if r_2 happens to be smaller than r_1] such that $f'(c) = 0$. But $f'(c) = 7c^6 + 30c^4 + 2 \geq 2$, so no such c exists. This contradiction implies that r_2 cannot exist, and this in turn shows that the assumption that there is another root r_2 distinct from r_1 is not tenable. We conclude that f has exactly one zero, so the given equation has exactly one root.

28. Let $f(x) = \sin x$. If $a = b$, then $|f(a) - f(b)| = |\sin a - \sin b| = 0$ and $|a - b| = 0$. So $|\sin a - \sin b| = |a - b|$. Next, we assume that $a < b$. The function f is continuous on $[a, b]$ and differentiable on (a, b). Using the Mean Value Theorem, we see that there exists a number c in (a, b) such that $\frac{f(b) - f(a)}{b - a} = \frac{\sin b - \sin a}{b - a} = f'(c) = \cos c$. Thus, $\sin b - \sin a = (b - a)\cos c \Rightarrow |\sin b - \sin a| = |b - a||\cos c|$. But $|\cos c| \leq 1$ and so $|\sin b - \sin a| \leq |b - a|$ and, combined with the previous result, we see that the inequality holds for all real numbers a and b.