Calculus Exam-2 (106.12.05)

- **1. (15%) Find an equation of the tangent line to the curve** $y = \sin xy$ **at point** $(\frac{\pi}{2}, 1)$

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◦ Sol :

$$y' = \cos xy \cdot (y + xy') = y\cos xy + y'(x\cos xy) \;\Rightarrow\; y' = \frac{y\cos xy}{1 - x\cos xy} \;\Rightarrow\; y'\Big|_{(x=\frac{\pi}{2},y=1)} = 0$$

Hence the equation of the tangent line is given by

$$y - 1 = 0 \;\Rightarrow\; y = 1$$

- **2. (15%) Prove that** $f(x) = x^5 + 6x + 4$ **has exactly one zero in** $(-\infty, \infty)$

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◦ Sol :

Because $f(-1) = -3 < 0$ and $f(0) = 4 > 0$, according to Intermediate Value Theorem , there exists $k \in (-1, 0)$ so that $f(k) = 0$.

Now we assume that there is another zero $p \in (-\infty, \infty)$ so that $f(p) = 0$ but $k \neq p$.

According to Roll's Theorem , there exists $q \in (k, p)$ or (p, k) so that $f'(q) = 0$. But

$$f'(x) = 5x^4 + 6 > 0 \;\Rightarrow\; f'(x) \text{ can not be zero}$$

So it is a contradiction. Hence f has exactly one zero in $(-\infty, \infty)$.

- **3. (15%) Find** f **by solving the initial value problem**

$$f'(x) = 1 + 1/x^2 \quad ; \quad f(1) = 2$$

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◦ Sol :

$$f(x) = \int f'(x)dx = \int 1 + \frac{1}{x^2}\, dx = x - \frac{1}{x} + C \quad ; \quad f(1) = 1 - 1 + C = 2 \;\Rightarrow\; C = 2$$

Hence

$$f(x) = x - \frac{1}{x} + 2$$

- **4. (15%) Use the definition of definite integral to evaluate**

$$\int_{-1}^{3}(4-x^2)dx$$

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- Sol :

The definition of definite integral

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta x$$

Now for this case

$$\Delta x = \frac{3-(-1)}{n} = \frac{4}{n} \quad ; \quad c_k = -1 + \frac{4k}{n}$$

Hence

$$\begin{aligned}
\int_{-1}^{3}(4-x^2)dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f\left(-1+\frac{4k}{n}\right)\left(\frac{4}{n}\right) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left[4-(-1+\frac{4k}{n})^2\right]\left(\frac{4}{n}\right) \\
&= \lim_{n\to\infty}\left(\frac{4}{n}\right)\sum_{k=1}^{n}\left(3+\frac{8k}{n}-\frac{16k^2}{n^2}\right) \\
&= \lim_{n\to\infty}\left[\frac{4}{n}\sum_{k=1}^{n}3+\frac{32}{n^2}\sum_{k=1}^{n}k-\frac{64}{n^3}\sum_{k=1}^{n}k^2\right] \\
&= \lim_{n\to\infty}\left[\frac{4}{n}\cdot 3n+\frac{32}{n^2}\cdot\frac{(1+n)n}{2}-\frac{64}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty}\left[12+16+\frac{16}{n}-\frac{32}{3}\left(\frac{2n^3+3n^2+n}{n^3}\right)\right] \\
&= 12+16-\frac{32}{3}\cdot 2 = 28-\frac{64}{3} = \frac{20}{3}
\end{aligned}$$

- **5. (10%) Find the horizontal asymptotes of the graph of the function**

$$\boldsymbol{f(x) = \frac{3x}{\sqrt{x^2+1}}}$$

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○ Sol :

For $x > 0$ then $\sqrt{x^2} = x$ then we have

$$\lim_{x\to\infty} \frac{3x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{3}{\sqrt{1+\frac{1}{x^2}}} = 3$$

For $x < 0$ then $\sqrt{x^2} = |x| = -x$ then we have

$$\lim_{x\to-\infty} \frac{3x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{3}{-\sqrt{1+\frac{1}{x^2}}} = -3$$

Hence $y = 3$ and $y = -3$ are the horizontal asymptotes of the graph of the function.

- **6. (10%) Find the relative extrema of the function**

$$f(x) = x^3 - 3x^2 - 24x + 32$$

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○ Sol :

$$f'(x) = 3x^2 - 6x - 24 = 3(x-4)(x+2) \ \therefore \text{ if } f'(x) = 0 \ \Rightarrow \ x = -2 \text{ or } 4$$

Hence $x = -2, 4$ are the critical numbers of f. The value are $f(-2) = 60$ and $f(4) = -48$

$$f''(x) = 6x - 6 = 6(x-1) \ \Rightarrow \ f''(-2) = -18 < 0 \quad ; \quad f''(4) = 18 > 0$$

Hence f has a relative maximum at $(-2, 60)$ and a relative minimum at $(4, -48)$.

- **7. (10%) Determine where the graph of the function** $g(x) = x^3 - 6x^2 + 2x + 3$ **is concave upward and where it is concave downward. Also find all inflection points.**

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◦ Sol :

$$g'(x) = 3x^2 - 12x + 2 \ \Rightarrow \ g''(x) = 6x - 12 = 6(x-2)$$

$$\begin{cases} \text{if } x \in (2, \infty)\ ,\ g'' > 0 & \text{the graph is concave upward} \\ \text{if } x \in (-\infty, 2)\ ,\ g'' < 0 & \text{the graph is concave downward} \end{cases}$$

And $g''(x) = 0$ at $x = 2$, so it has an reflection point at $x = 2$.

- **8. (10%) Find**

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$$

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◦ Sol :

Let $u = \sqrt{x}$ then

$$u = \sqrt{x} \ \Rightarrow \ du = \frac{1}{2}x^{-1/2}dx = \frac{dx}{2\sqrt{x}} \ \Rightarrow \ 2du = \frac{dx}{\sqrt{x}}$$

Hence

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx = \int \cos u \cdot 2du = 2\sin u + C = 2\sin\sqrt{x} + C$$
