Chapter 15: Vector Analysis

§15.1 Vector Fields.

<u>Def'n</u>: Let M and N be functions of two variables x and y, defined on a planar region R. The function $\vec{F}$ defined by $\vec{F}(x,y) = M\vec{i} + N\vec{j}$ is called a vector field over R.

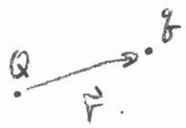
Let M, N and P be functions of three variables x, y and z, defined on a solid region Q in space. The function $\vec{F}$ defined by $\vec{F}(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ is called a vector field over Q.

<u>Examples</u>: (1) Gradient Fields $\vec{\nabla} f(x,y)$, $\vec{\nabla} f(x,y,z)$.

(2) Velocity Fields (Accelaration Fields)

(3) Gravitation Field $\vec{F} = -\dfrac{GMm}{\|\vec{r}\|^3}\vec{r}$.

(4) Electric Force Fields: $\vec{F} = \dfrac{cQq}{\|\vec{r}\|^3}\vec{r}$



<u>Example</u>: Sketch some vectors in the vector field given by $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$.

<u>Def'n</u>: A vector field $\vec{F}$ is called conservative if there exists a differentiable function f s.t. $\vec{F} = \vec{\nabla} f$. The function f is called the potential function for $\vec{F}$.

(It is preferred $\vec{F} = -\vec{\nabla} f$ in physics; if ~~the~~ force is conservative, then we have the conservation law of total energy.)

How to check a vector field is conservative or not?

<u>Theorem 15.1</u> Let M and N have continuous 1st partial derivatives on an open disk R. The vector field given by $\vec{F}(x,y) = M\vec{i} + N\vec{j}$ is conservative if and only if $N_x = M_y$

Example: Find a potential function for $\vec{F}(x,y) = 2xy\,\vec{i} + (x^2-y)\,\vec{j}$.
(Practise Theorem 15.1 first.)

Def'n: The curl of $\vec{F}(x,y,z) = M\vec{i} + N\vec{j} + P\vec{k}$ is

$$\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = (P_y - N_z)\vec{i} - (P_x - M_z)\vec{j} + (N_x - M_y)\vec{k}.$$

$$\parallel$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix}$$

If $\text{curl}\,\vec{F} = \vec{0}$, then $\vec{F}$ is said to be irrotational.

Continue our previous question: When a vector field is conservative?

Theorem 15.2: Suppose M, N, and P have continuous 1st partial derivatives in an open sphere Q in space. The vector field given by $\vec{F}(x,y,z) = M\vec{i} + N\vec{j} + P\vec{k}$ is conservative if and only if $\text{curl}\,\vec{F} = \vec{0}$.

Example: Find a potential function for $\vec{F}(x,y,z) = 2xy\,\vec{i} + (x^2+z^2)\vec{j} + 2yz\,\vec{k}$. (Practise Theorem 15.2 first!)

Def'n: The divergence of $\vec{F}(x,y) = M\vec{i} + N\vec{j}$ is

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = M_x + N_y$$

The divergence of $\vec{F}(x,y,z) = M\vec{i} + N\vec{j} + P\vec{k}$ is

$$\text{div}\,\vec{F} = \vec{\nabla}\cdot\vec{F} = M_x + N_y + P_z.$$

If $\text{div}\,\vec{F} = 0$, then $\vec{F}$ is said to be divergence free.

Theorem 15.3: If $\vec{F}(x,y,z) = M\vec{i} + N\vec{j} + P\vec{k}$ is a vector field, and M, N, and P have continuous 2nd partial derivatives, then

$$\operatorname{div}(\operatorname{curl}\vec{F}) = 0.$$

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§15.2 Line Integrals.

<u>Recall:</u> ① A curve, $\vec{r} = x(t)\vec{i} + y(t)\vec{j}$ or $x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, is smooth (or piecewise smooth) if ...

② <u>Orientation</u>.

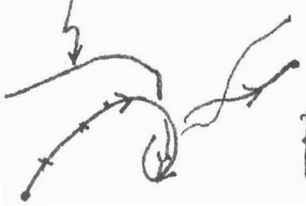
<u>Def'n</u>: If f is defined in a region containing a smooth curve C of finite length, then

$$\int_C f(x,y)\,ds = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i)\,\Delta s_i.$$

or

$$\int_C f(x,y,z)\,ds = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta s_i.$$

graph of f



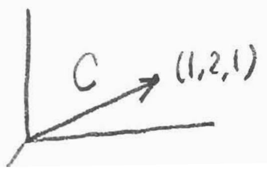
provided this limit exists.

<u>Theorem 15.4</u>: Let f be continuous in a region containing a smooth curve C. If C is given by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, where $a \le t \le b$, then

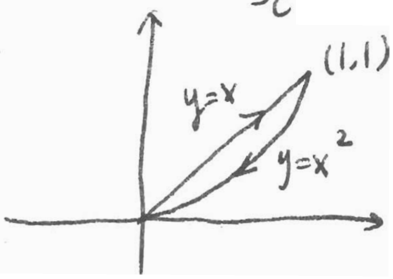
$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

(It is similar in 3D).

<u>Example</u>: Evaluate $\int_C (x^2 - y + 3z)\,ds$ where C is a line segment.



Example: Evaluate $\int_C x\,ds$ where C is the piecewise smooth curve



Defn: Let $\vec{F}$ be a continuous vector field defined on a ~~smooth~~ smooth curve C given by $\vec{r}(t)$, $a \le t \le b$. The line integral of $\vec{F}$ on C is given by

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot \vec{T}\,ds$$

$$= \int_C \vec{F}\cdot \vec{r}\,'(t)\,dt$$

$\because ds = \|\vec{r}\,'(t)\|\,dt$, $\vec{T} = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$

Example: Let $\vec{F}(x,y) = y\vec{i} + x^2\vec{j}$ and evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ for each parabolic curve

C_1: $\vec{r}_1(t) = (4-t)\vec{i} + (4t-t^2)\vec{j}$, $0 \le t \le 3$

C_2: $\vec{r}_2(t) = t\vec{i} + (4t-t^2)\vec{j}$, $1 \le t \le 4$.

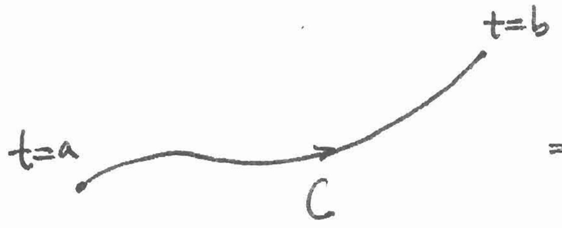
Remark: ~~Graphically~~ Geometrically, C_1 and C_2 have the same graph but different orientations. For describing such curves, we ~~would~~ usually write $C_2 = -C_1$ (or $C_1 = -C_2$ whichever you prefer) So

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{-C_1} \vec{F}\cdot d\vec{r}$$

Meanwhile, our computation show $\int_{C_2} \vec{F}\cdot d\vec{r} = -\int_{C_1} \vec{F}\cdot d\vec{r}$

So, this example strongly suggests

$$\int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$$



$C: \vec{r}(t)$ for $a \le t \le b$

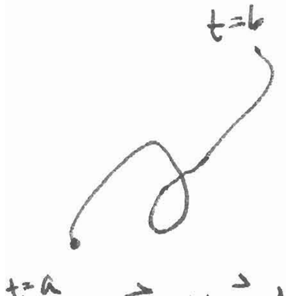
$\Rightarrow -C: \vec{r}(\cancel{\theta})$ for $a \le \overset{t}{\cancel{a}} \le b$

~~where $t = \theta$~~ ~~$t+(b-a)$~~ $b+a-t$.

$$\int_{-C} \vec{F}\cdot d\vec{r} = \int_a^b (\vec{F}\cdot\vec{r}\,')\cdot(-1)\,dt$$

$$= -\int_a^b \vec{F}\cdot\vec{r}\,'\,dt = -\int_C \vec{F}\cdot d\vec{r}.$$

A <u>neat form of line integrals</u>:



$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}\cdot\vec{r}\,'(t)\,dt$$

$$= \int_a^b Mx' + Ny'\,dt.$$

$$= \int_C M\,dx + N\,dy.$$

differential

$dx = x'\,dt$
$dy = y'\,dt.$

$\vec{F} = M\vec{i} + N\vec{j}$

2D

Similar, in 3D

$$\int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy + P\,dz.$$

if $\vec{F} = M\vec{i} + N\vec{j} + P\vec{k}$

<u>Example</u> Let C be the circle given by $\vec{r}(t) = 3\cos t\,\vec{i} + 3\sin t\,\vec{j}$, $0 \le t \le 2\pi$. Evaluate $\displaystyle\int_C y^3\,dx + (x^3 + 3xy^2)\,dy.$

§15.3 Conservative Vector Fields and Independence of Path.

Theorem 15.5: Let C be a piecewise smooth curve lying in an open region R and given by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, $a \le t \le b$. If $\vec{F}(x,y) = M\vec{i} + N\vec{j}$ is conservative in R, and M and N are continuous in R, then

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{\nabla} f\cdot d\vec{r} = f(x(b),y(b)) \quad f(x(a),y(a))$$

where f is a potential function of $\vec{F}$.

Pf: Since f is a potential of $\vec{F}$, we write

$$\vec{F} = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}.$$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_a^b f_x(x(t),y(t))\,x'(t) + f_y(x(t),y(t))\,y'(t)\,dt.$$

$$= \int_a^b \frac{d}{dt} f(x(t),y(t)) = f(x(b),y(b)) - f(x(a),y(a)).$$

In particular, if C is a closed curve, then

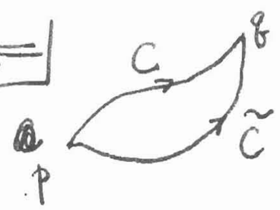
$$\int_C \vec{F}\cdot d\vec{r} = 0.$$

Read Example 1:

Example: Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is a piecewise smooth curve from $(1,1,0)$ to $(0,2,3)$ and $\vec{F}(x,y,z) = 2xy\,\vec{i} + (x^2+z^2)\,\vec{j} + 2yz\,\vec{k}$.

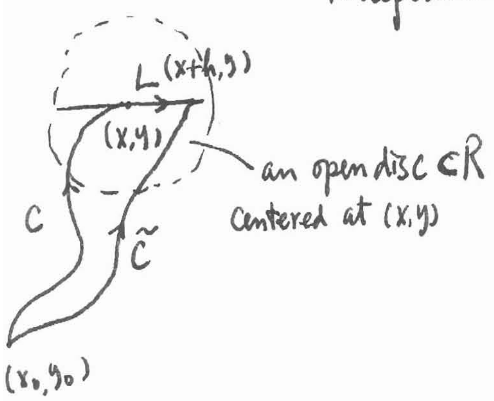
<u>Theorem 15.6</u>: If $\vec{F}$ is continuous on an open connected region, then the line integral $\int_C \vec{F}\cdot d\vec{r}$ is independent of path. if and only if $\vec{F}$ is conservative.

<u>pf</u>: $\Leftarrow\rfloor$ This is Theorem 15.5.



$\Rightarrow\rfloor$ Let $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$, and (x_0,y_0) be a (fixed) point in the region R. Let (x,y) be an arbitrary point in R, and suppose C is a curve in R connecting (x_0,y_0) and (x,y), as show in Figure.

Define $f(x,y) := \int_C \vec{F}\cdot d\vec{r}$. Since the line integral is independent of paths, f is well-defined.



$$
\begin{aligned}
& f(x+h,y) - f(x,y) \\
&= \int_{\tilde{C}} \vec{F}\cdot d\vec{r} - \int_C \vec{F}\cdot d\vec{r} \\
&= \int_L \vec{F}\cdot d\vec{r} = \int_L M\,dx + N\,dy \\
&= \int_L M\,dx
\end{aligned}
$$

$\Rightarrow \dfrac{f(x+h,y)-f(x,y)}{h} = \dfrac{1}{h}\int_L M\,dx \longrightarrow M(x,y)$ as $h \to 0$

by MVT & since M is conti.

Similarly, $\dfrac{f(x,y+k)-f(x,y)}{k} \Rightarrow N(x,y)$ as $k \to 0$

$\therefore f$ has the 1st partial derivatives, and

$f_x = M,\ f_y = N$ (since both of them are conti. $\Rightarrow f$ is diff) $\Rightarrow f$ is a potential function of $\vec{F}$

$\therefore$ $\vec{F}$ is conservative.

<u>Theorem 15.7</u>: Let $\vec{F}(x,y,z) = M\vec{i} + N\vec{j} + P\vec{k}$ have continuous 1st partial derivatives in an open connected region R, and let C be a piecewise smooth curve in R. TFAE:

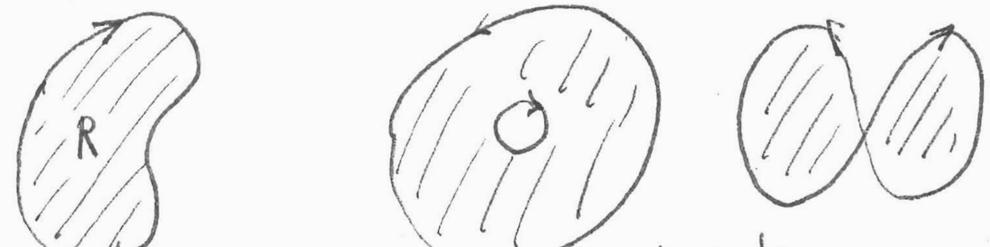
1. $\vec{F}$ is conservative
2. $\int_C \vec{F}\cdot d\vec{r}$ is independent of path
3. $\int_C \vec{F}\cdot d\vec{r} = 0$ for every <u>closed curve</u> C in R

$\vec{r}(a) = \vec{r}(b)$ supposed

<u>Example</u>: Evaluate $\int_{C_1} \vec{F}\cdot d\vec{r}$, where $\vec{F}(x,y) = (y^3+1)\vec{i} + (3xy^2+1)\vec{j}$ and C_1 is the semicircular path from $(0,0)$ to $(2,0)$.

<u>§15.4</u> Green's Theorem

<u>Defn</u>: Let C be a planar curve, given by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ for $a \le t \le b$. C is simple if it does not cross itself, that is $\vec{r}(c) \ne \vec{r}(d)$ for all $c, d \in (a,b)$. A planar region R is simply connected if its boundary consists of <u>one</u> simple closed curve.



simply connected

not simply connected.

Theorem 15.8: Green's Theorem

Let R be a simply connected region with a piecewise smooth boundary C, oriented counterclockwise. If M and N have continuous partial derivatives in an open region containing R, then

$$\iint_R (N_x - M_y)\,dA = \int_C M\,dx + N\,dy.$$

Remarks: ① Green's Theorem is a formal generalization of integration by parts.

$$\int_C M\,dx + N\,dy = \int_{\partial R} M\,dx + N\,dy$$

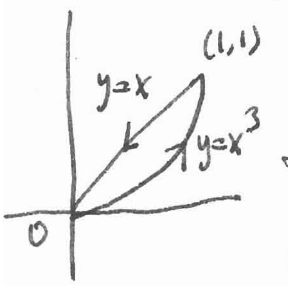
$$= \iint_R d(M\,dx + N\,dy) = \iint_R (N_x - M_y)\,dx \wedge dy$$

$$\parallel$$

$$\iint_R (N_x - M_y)\,dA.$$

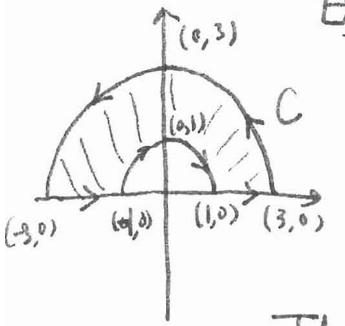
② Very complicated to prove, due to the pathology of regions.

Example: Use Green's Theorem to evaluate the line integral



$\int_C y^3\,dx + (x^3 + 3xy^2)\,dy$ where C is a path from $(0,0)$ to $(1,1)$ along the graph of $y = x^3$ and from $(1,1)$ to $(0,0)$ along the graph of $y = x$

Remark: Use the def'n of line integral to evaluate the line integral again!



Example: Evaluate $\int_C (\tan^{-1}x + y^2)\,dx + (e^y - x^2)\,dy$

Theorem 15.9: If R is a planar region bounded by a piecewise smooth simple closed curve C, oriented counterclockwise, then

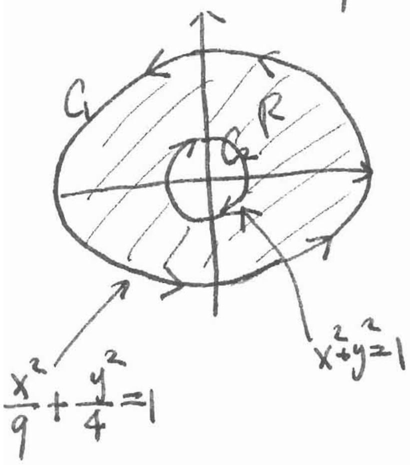
$$\text{Area}(R) = \frac{1}{2}\int_C x\,dy - y\,dx.$$

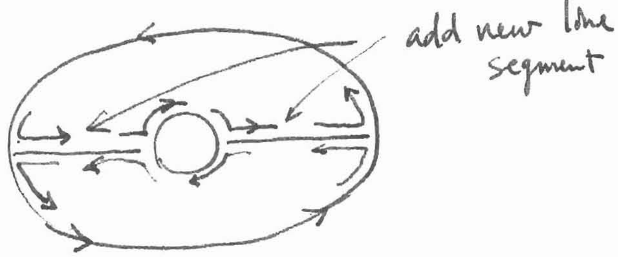
(Direct application of Green's Theorem)

Example Let R be the region shown on the left

Evaluate $\int_C 2xy\,dx + (x^2 + 2x)\,dy$, where $C = C_1 + C_2$.

is the boundary of R.



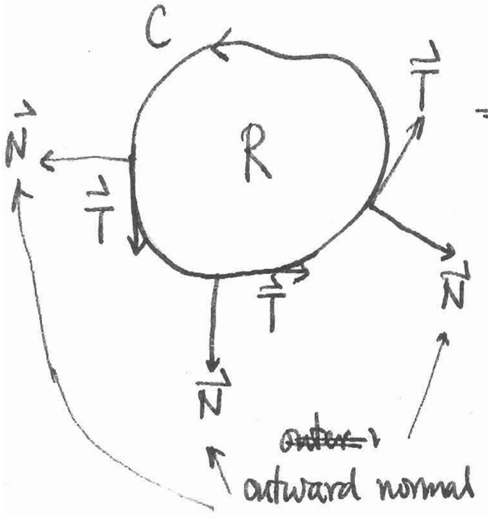


Alternative forms of Green's Theorem

① Suppose $\vec{F}(x,y,z) = M\vec{i} + N\vec{j} + 0\vec{k}$: a vector field in the xy-plane

then $\int_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\cdot\vec{k}\,dA$

— extending to surfaces in 3D as the Stokes' Theorem §15.8



③ Let C be given by $\vec{r}(s) = x(s)\vec{i} + y(s)\vec{j}$, where s is the arc length parameter for C.

$\Rightarrow \vec{r}'(s) = \vec{T}(s) = x'(s)\vec{i} + y'(s)\vec{j}$

Let $\vec{N}(s) = y'(s)\vec{i} - x'(s)\vec{j}$.

$$\Rightarrow \int_C \vec{F}\cdot\vec{N}\,ds = \int_C M\,dy - N\,dx \quad (\text{for } \vec{F} = M\vec{i} + N\vec{j})$$

$$= \iint_R M_x + N_y \, dA$$

$$= \iint_R \operatorname{div}\vec{F}\, dA.$$ — Extending to 3D as the Divergence Theorem §§15.7, 8.