Chapter 14. Multiple Integration.

§14.1 Iterated Integrals and Area in the plane.

Given a function $f(x,y)$, $\int f(x,y)\,dx$ = all functions whose first partial derivative w.r.t. x is $f(x,y)$. Similar for $\int f(x,y)\,dy$.

Example: $\int 2xy\,dx$, $\int 2xy\,dy$.

So, we may use the FTC to compute integrals like

Example: $\int_1^x (2x^2y^{-2}+2y)\,dy$.

Finally, ~~the~~ ~~int~~ we calculate iterated integrals.

$$\int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx$$

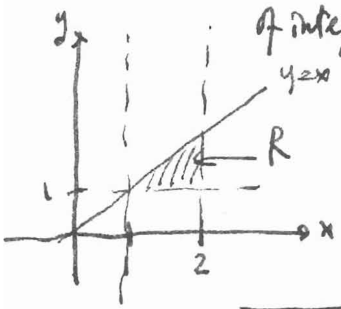
or

$$\int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy.$$

Example: $$\int_1^2 \left(\int_1^x (2x^2y^{-2}+2y)\,dy\right) dx$$

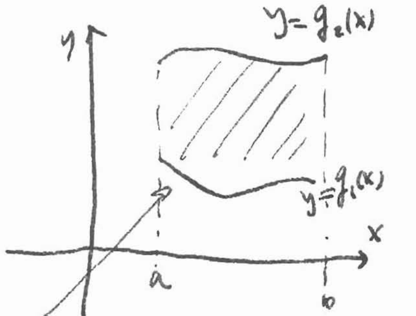
outside limits of integration

inside limits of integration



from these two informations, we can construct the so-called region of integration R.

Derived from



vertically simple

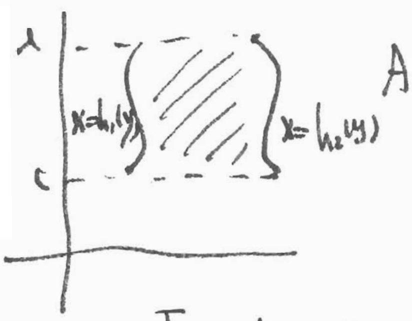
$$A = \int_a^b g_2(x) - g_1(x)\,dx$$

$$= \int_a^b \left(\int_{g_1(x)}^{g_2(x)} dy\right) dx$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$$

meaning that area of (some) planar regions can be expressed in terms of iterated integrals

Similarly, for some planar regions (horizontally simple)



$$A=\int_c^d\int_{h_1(y)}^{h_2(y)} dx\,dy$$

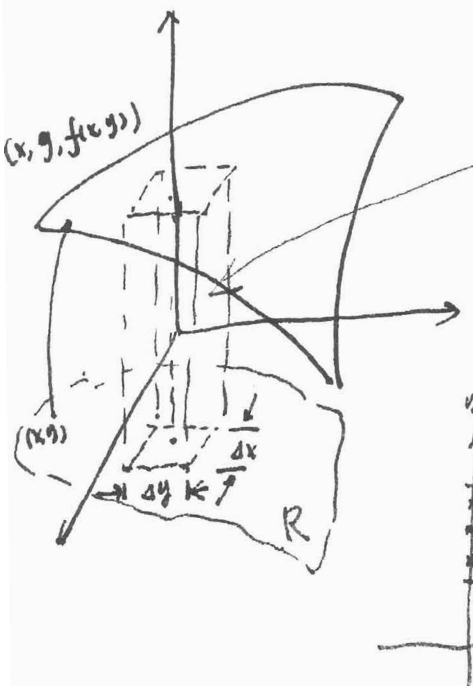
Example: Use an iterated integral to find the area of the region bounded by the graphs of $f(x)=\sin x$, $g(x)=\cos x$. between $x=\pi/4$ and $x=5\pi/4$.

Example: Sketch the region whose area is represented by the integral $\int_0^2\int_{y^2}^4 dx\,dy$. Then find another iterated integral using the order $dy\,dx$ to represent the same area and show that both integrals yield the same value.

Example: Find the area of the region R that lies below the parabola $y=4x-x^2$ above the x-axis, and above the line $y=-3x+6$.

---

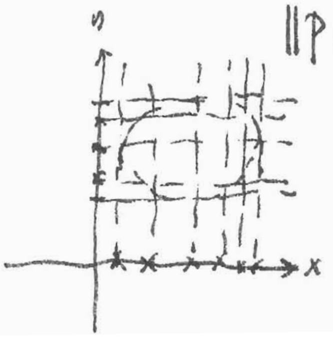
§ 14.2 Double integrals and Volume



$f(c,d)\,\Delta x\,\Delta y$

$$\sum f(c_i,d_j)\,\Delta x_i\,\Delta y_j \xrightarrow{\|P\|\to 0} V$$

In this case, we call that $f(x,y)$ is integrable over R, and the limit V is denoted by $\iint_R f(x,y)\,dA$.



$\|P\| := \max\limits_{\substack{1\le i\le m\\ 1\le j\le n}}$ of the length of the longest diagonal of the rectangle $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$

Remarks: (1) If $f(x,y)$ is integrable over R and $f(x,y) \geq 0$ $\forall (x,y) \in R$, then the volume of the solid region that lies above R and below the graph of f is ~~denoted by~~ defined as $\iint_R f(x,y)\,dA$.

(2) If $f(x,y)$ is continuous over a closed and bounded region R, then f is integrable over R.

Theorem 14.1: Let f and g be continuous over a closed, bounded planar region R, and ~~let~~ let c be a constant.

1. $\iint_R c\,f(x,y)\,dA = c \iint_R f(x,y)\,dA$.
2. $\iint_R \left(f(x,y) \pm g(x,y)\right) dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$.
3. $\iint_R f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$
4. $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$ if $f(x,y) \geq g(x,y)$
5. $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$,

   where $R = R_1 \cup R_2$, R_1 and R_2 are two nonoverlapping regions.

Conceptually, double integrals are clear. But how to compute them?

Theorem 14.2 (Fubini)

Let f be continuous on a planar region R.

1. If R is defined by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(y)$ where g_1 and g_2 are continuous on $[a,b]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

2. If R is defined by $c \le y \le d$; $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous on $[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Example: Find the volume of the solid region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the xy-plane.

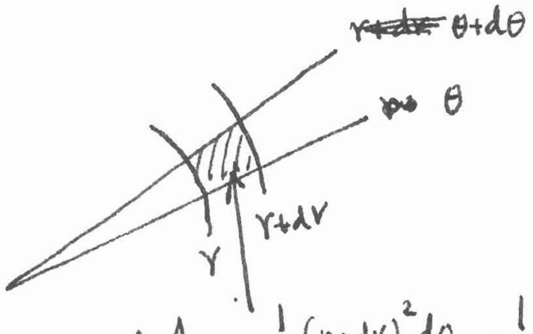
Example: Find the volume of the solid region R bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the plane $z = 1 - y$

✓ Example: Find the volume of the solid region R bounded by the surface $f(x,y) = e^{-x^2}$ and the planes $z=0$, $y=0$, $y=x$ & $x=1$.

---

§ 14.3 Change of Variables: Polar Coordinates.

Partitions in r- and θ- directions



$$dA = \frac{1}{2}(r+dr)^2 d\theta - \frac{1}{2} r^2 d\theta$$
$$= r\,dr\,d\theta$$

Riemann sum : $\sum f(r_i, \theta_j)\, r_i\, \Delta r_i\, \Delta \theta_j$

Theorem 14.3: Let R be a planar region consisting of all points $(x,y) = (r\cos\theta, r\sin\theta)$ satisfying the conditions $0 \le g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, where $0 \le (\beta - \alpha) \le 2\pi$. If g_1 and g_2 are continuous on

$[\alpha, \beta]$ and f is continuous on R, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

<u>Example</u>: Use polar coordinates to find the volume of the solid region bounded above by the hemisphere

$$z = \sqrt{16 - x^2 - y^2}$$

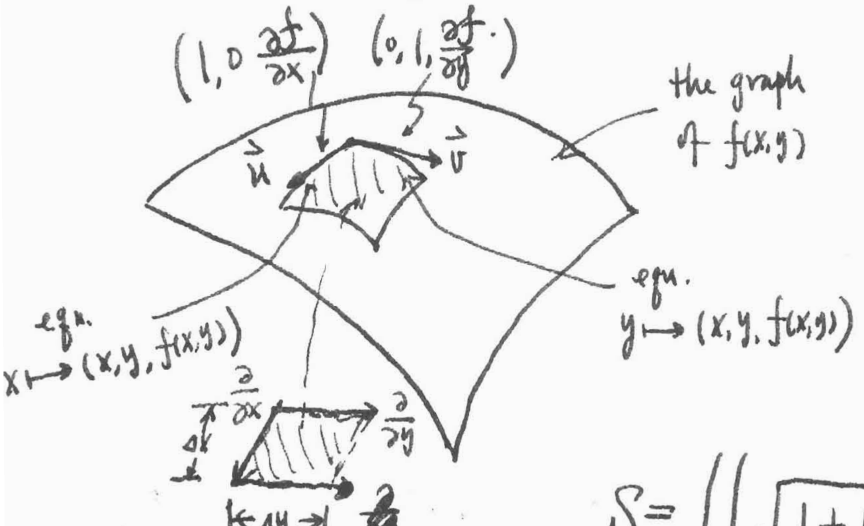
and below by the circular region R given by $x^2 + y^2 \leq 4$.

<u>Example</u>: Find the area of the region bounded above by the spiral $r = \pi/3\theta$ and below by the polar axis

<u>Example</u>: Use a double integral to find the area enclosed by the graph of $r = 3\cos 3\theta$.

---

§14.5 <u>Surface Area</u>



$\because$ tangent plane approximates the surface locally,

$$\therefore \| \Delta x \vec{u} \times \Delta y \vec{v} \| = \sqrt{f_x^2 + f_y^2 + 1}\; dA =: dS$$

$$S = \iint_R \sqrt{1 + \|\vec{\nabla} f\|^2}\; dA.$$

provided f, f_x, f_y are continuous on a closed region R.

Example: Find the surface area of the portion of the plane $z = 2 - x - y$ that lies above the circle $x^2+y^2 \le 1$ in the first quadrant.

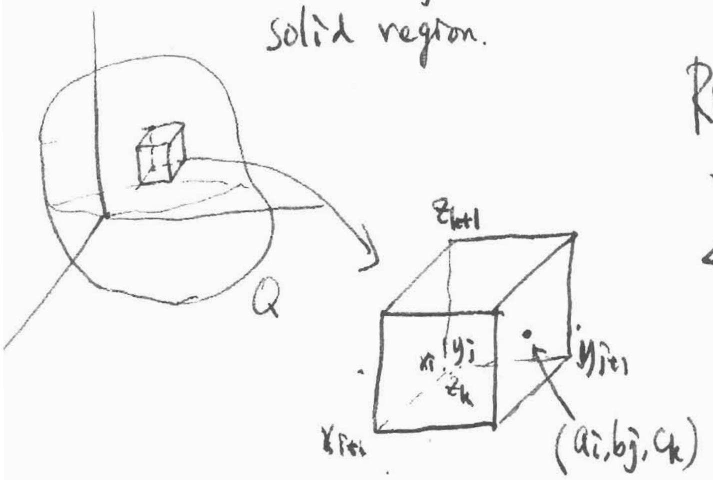
Example: Find the area of the portion of the surface $f(x,y) = 1 - x^2 + y$ that lies above the triangular region with vertices $(1,0,0)$, $(0,-1,0)$, and $(0,1,0)$.

Example: Find the surface area of the paraboloid $z = 1 + x^2 + y^2$ that lies above the unit circle.

---

§14.6: Triple Integrals and Applications.

Let $f(x,y,z)$ be a ~~continuous~~ function defined on a solid region.



Riemann sum

$$\sum f(a_i, b_j, c_k)\, \underset{x_{i+1}-x_i}{\underset{\parallel}{\Delta x_i}}\, \overset{y_{i+1}-y_i}{\overset{\parallel}{\Delta y_j}}\, \underset{z_{k+1}-z_k}{\underset{\parallel}{\Delta z_k}}$$

$$\xrightarrow{\|P\| \to 0} L \text{, if possible}$$

$\|P\|$: max of ~~each~~ the diameter of each cube.

$\Rightarrow$ f is integrable over Q.

L is called the triple integral of f over Q, denoted by $\iiint_Q f(x,y,z)\,dV$.

Especially, if $f(x,y,z) = 1$, then $\iiint_Q dV =$ the volume of Q

Like double integrals, we first practice computations of iterated integrals.

<u>Example</u>: Evaluate the triple iterated integral.

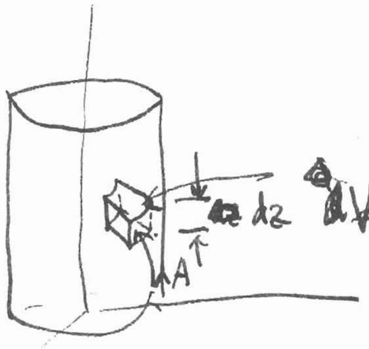
$$\int_0^2 \int_0^x \int_0^{x+y} e^x (y+2z)\, dz\, dy\, dx.$$

<u>Example</u>: Find the volume of the ellipsoid given by $4x^2+4y^2+z^2=16$.

<u>Example</u>: Evaluate $\displaystyle\int_1^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \int_1^3 \sin y^2\, dz\, dy\, dx$.

---

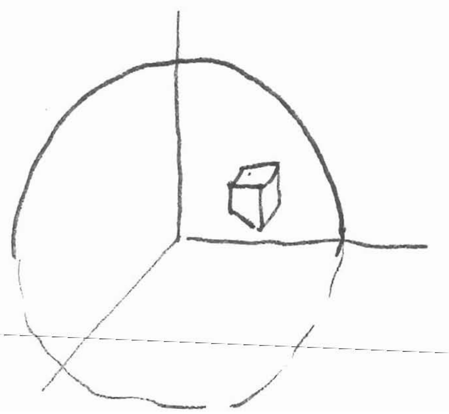
§ 14.7 Triple Integrals in Cylindrical and Spherical Coordinates.

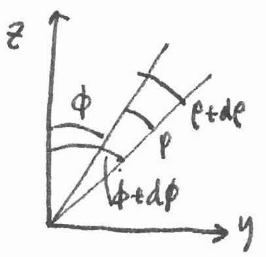


~~$dV = \Delta A \Delta z = r \Delta r \Delta\theta \Delta z$~~

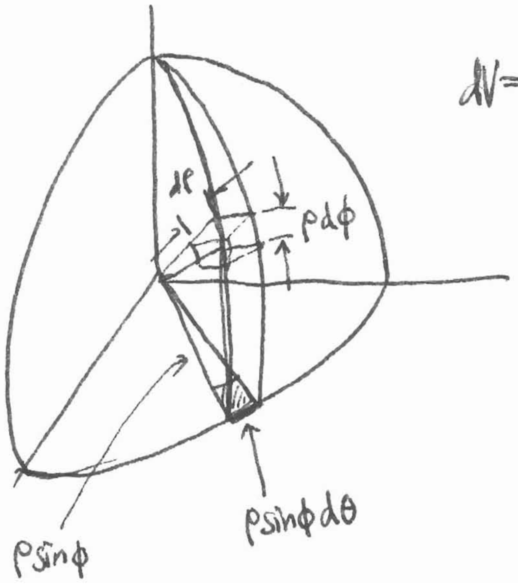
$$dV = A\,dz = \left.\begin{cases} r\,dr\,d\theta\,dz \\ r\,dr\,dz\,d\theta \\ \vdots \end{cases}\right\} 6$$

<u>Example</u>: Find the volume of the solid region Q cut from the sphere $x^2+y^2+z^2=4$ by the cylinder $r=2\sin\theta$.





or this is better to comprehend.



$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\left.\begin{matrix} d\rho \, d\theta \, d\phi \\ \vdots \end{matrix}\right\} 6.$$

Example: Find the volume of the solid region Q bounded below by the upper nappe of the cone $z^2 = x^2 + y^2$ and above by the sphere $x^2+y^2+z^2=9$.

---

§14.8: Change of Variables: Jacobian

Def'n: If $x = g(u,v)$ and $y = h(u,v)$, then the Jacobian of x and y with respect to u and v, denoted by $\frac{\partial(x,y)}{\partial(u,v)}$ is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Example: ① $x = r\cos\theta$, $y = r\sin\theta \Rightarrow \frac{\partial(x,y)}{\partial(r,\theta)} = ?$

② $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi \Rightarrow \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = ?$

Theorem 14.5: Let R and S be regions in the xy- and uv-planes that are related by the equations $x = g(u,v)$ and $y = h(u,v)$ such that each point in R is the image of a unique point in S. If f is continuous on R, g and h have continuous partial derivatives on S, and $\dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$ on S,

then
$$\iint_R f(x,y)\,\underbrace{dx\,dy}_{\text{or } dA} = \iint_S f(g(u,v),h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \underbrace{du\,dv}_{\text{or } dA'}$$

more appropriately

Remark: Students should read the proof.

Example: Let R be the region bounded by the lines
$x-2y=0$, $x-2y=-4$, $x+y=4$, $x+y=1$.
Evaluate the double integral $\displaystyle\iint_R 3xy\,dA$

Example: Let R be the region bounded by the square with vertices $(0,1)$, $(1,2)$, $(2,1)$ and $(1,0)$. Evaluate the integral
$$\iint_R (x+y)^2 \sin^2(x-y)\,dA.$$