Chapter 12: Vector-Valued Functions.

§12.1 Space Curves and Vector-Valued Functions.

Like plane curves, a space curve is defined to be parametric equations $x=f(t)$, $y=g(t)$ and $z=h(t)$ for $t \in$ some interval I. Notions, such as graph, smoothness, piecewise smoothness are similarly defined for space curves.

Defn: A function of the form

$$\vec{r}(t) := f(t)\,\vec{i} + g(t)\,\vec{j}$$

or

$$\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$$

is a vector-valued function, where the component functions f, g and h are real-valued functions of the parameter t. Vector-valued functions are sometimes denoted as $\vec{r}(t) = (f(t), g(t))$ or $\vec{r}(t) = (f(t), g(t), h(t))$.

Remark: The so-called vector-valued functions ~~vector form~~ reinterpret ~~are~~ parametric equations ~~~~ in the form of vectors.

① Like vectors, we can add, substract, take inner product, cross product (if allowed) two vector-valued functions, as well take scalar multiplization.

② Like parametric equations, we may discuss continuity, differentiability of vector-valued functions.

Defn: If $\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j}$ (or $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$) is a vector-valued functions, then

$$\lim_{t\to a} \vec{r}(t) = \left[\lim_{t\to a} f(t)\right]\vec{i} + \left[\lim_{t\to a} g(t)\right]\vec{j}$$

$$\left(\text{or } \lim_{t\to a} \vec{r}(t) = \left[\lim_{t\to a} f(t)\right]\vec{i} + \left[\lim_{t\to a} g(t)\right]\vec{j} + \left[\lim_{t\to a} h(t)\right]\vec{k}\right).$$

<u>Remark</u>: If the vector on the right-hand side exists, say L, then

$$\lim_{t\to a} \vec{r}(t) = \vec{L} \quad \equiv \quad \lim_{t\to a} \|\vec{r}(t) - \vec{L}\| = 0.$$

<u>Def'n</u>: A vector-valued function $\vec{r}$ is continuous at the point given by $t=a$ if $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$. A vector-valued function $\vec{r}$ is continuous on an interval if it is continuous at every point in the interval.

<u>Examples</u>: $\vec{r}(t) := t\vec{i} + a\vec{j} + (a^2 - t^2)\vec{k}$, where a is a constant.

<u>Remark</u>: $\vec{r}(t)$ is continuous at the point given by $t=a$ iff each component function of $\vec{r}(t)$ is continuous at $t=a$.
$\vec{r}(t)$ is continuous on an interval iff each component function is continuous on the interval.

<u>§12.2</u> Differentiation and integration of vector-valued functions.

<u>Def'n</u>: The derivative of a vector-valued function $\vec{r}$ is defined by

$$\vec{r}'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

for all t for which the limit exists. If $\vec{r}'(c)$ exists, then $\vec{r}$ is differentiable at c. If $\vec{r}'(c)$ exists for all c in an open interval I, then $\vec{r}$ is differentiable on the interval I

Remarks (1) The derivative of $\vec{r}(t)$ is constituted by the derivatives of the component functions of $\vec{r}(t)$. (Theorem 12.1)

(2) $\vec{r}(t)$ is differentiable $\equiv$ each component function is differentiable.

(3) If possible, we may define higher-order derivatives of $\vec{r}(t)$.

Examples: (1) $\vec{r}(t) = t^2\vec{i} - 4\vec{j}$, (2) $\vec{r}(t) = \frac{1}{t}\vec{i} + \ln t\,\vec{j} + e^{2t}\vec{k}$

Theorem 12.2: Let $\vec{r}$ and $\vec{u}$ be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

(1) $D_t(c\vec{r}(t)) = c\vec{r}\,'(t)$, (2) $D_t(\vec{r}(t) \pm \vec{u}(t)) = \vec{r}\,'(t) + \vec{u}\,'(t)$

(3) $D_t(f(t)\vec{r}(t)) = f'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$

(4) $D_t(\vec{u}(t)\cdot\vec{r}(t)) = \vec{u}\,'(t)\cdot\vec{r}(t) + \vec{u}(t)\cdot\vec{r}\,'(t)$

(5) $D_t(\vec{r}(t)\times\vec{u}(t)) = \vec{r}\,'(t)\times\vec{u}(t) + \vec{r}(t)\times\vec{u}\,'(t)$

(6) $D_t(\vec{r}(f(t))) = \vec{r}\,'(f(t))\,f'(t)$,

(7) If $\vec{r}(t)\cdot\vec{r}(t) = c$, then $\vec{r}\,'(t) \perp \vec{r}(t)$.

Motivated by $\frac{d}{dt}\int f(t) = f(t)$,

Defn: (1) If $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j}$, where f and g are continuous on $[a,b]$, then the indefinite integral (antiderivative) of $\vec{r}$ is

$$\int \vec{r}(t)\,dt = \left(\int f(t)\,dt\right)\vec{i} + \left(\int g(t)\,dt\right)\vec{j}.$$

and its definite integral over the interval $a \le t \le b$ is

$$\int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j}.$$

(2) For vector-valued functions $\vec{r}(t) \in \mathbb{R}^3$, the indefinite integral and the definite integral over the interval $[a,b]$ are similarly defined (the student are strongly encouraged to write them down).

Example: Find the antiderivative of $\vec{r}\,'(t) = \cos 2t\,\vec{i} - 2\sin t\,\vec{j} + \frac{1}{1+t^2}\vec{k}$ that satisfies the initial condition $\vec{r}(0) = 3\vec{i} - 2\vec{j} + \vec{k}$.

§12.4: Tangent Vectors and Normal Vectors.

Def'n: Let C be a smooth curve represented by $\vec{r}$ on an open interval I. The unit tangent vector $\vec{T}(t)$ at t is defined to be

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}, \quad \vec{r}\,'(t) \neq \vec{0}.$$

Example: Find $\vec{T}(t)$ and then find a set of parametric equations for the tangent line to the helix given by $\vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + t\vec{k}$ at the point corresponding to $t = \pi/4$.

Def'n: As assumed in the preceding def'n. If $\vec{T}\,'(t) \neq 0$, then the principal unit normal vector at t is defined to be

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\|\vec{T}\,'(t)\|}$$

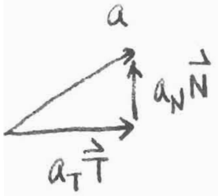
Rmk: $\vec{T} \perp \vec{T}\,'$ $\Rightarrow \vec{T} \perp \vec{N}$.

Example: Find the principal unit normal vector for the helix given by $\vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + t\vec{k}$.

Def'n: If $\vec{r}(t)$ is the position vector for a smooth curve C, then $\vec{r}\,'(t)$, if exists, is called the velocity vector $\vec{v}$, and, furthermore, if $\vec{r}\,''(t)$ exists, then it is called the acceleration vector $\vec{a}$. $\|\vec{v}\|$ is called the speed.

Theorem 12.4 If $\vec{r}(t)$ is the position vector for a smooth curve C and $\vec{N}(t)$ exists, then the acceleration vector $\vec{a}(t)$ lies in the plane determined by $\vec{T}(t)$ and $\vec{N}(t)$.

$\left(\because\ \vec{v} = \vec{r}'(t) = \|\vec{r}'(t)\|\,\vec{T}(t)\right)$



$$\therefore\ \vec{a} = \vec{v}' = \left(\frac{d}{dt}\|\vec{r}'(t)\|\right)\vec{T}(t) + \|\vec{r}'(t)\|\,\|\vec{T}'(t)\|\,\vec{N}(t)$$

diff when $\vec{r}'(t) \neq 0$

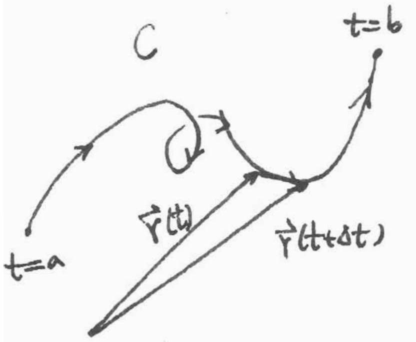
Let $a_T := \dfrac{d}{dt}\|\vec{r}'(t)\|$ (if exists) $= \vec{a}\cdot\vec{T} = \vec{a}\cdot\dfrac{\vec{v}}{\|\vec{v}\|}$

$$0 < a_N := \|\vec{r}'(t)\|\,\|\vec{T}'(t)\| = \|\vec{v}\|\,\|\vec{T}'(t)\| = \sqrt{\|\vec{a}\|^2 - a_T^2}$$

$$= \frac{\|\vec{v}\times\vec{a}\|}{\|\vec{v}\|}$$

§12.5 Arc Length and Curvature.

We have learnt how to compute the arc length of a smooth curve C represented by parametric equations $x = f(t)$, $y = g(t)$.
We may extend the computation to space curves and write the process in terms of vector-valued functions.



$$s = \int_a^b \|\vec{r}'(t)\|\,dt \longleftarrow \text{Theorem 12.6}$$

Example: Find the length of one turn of the helix given by $\vec{r}(t) = b\cos t\,\vec{i} + b\sin t\,\vec{j} + \sqrt{1-b^2}\,t\,\vec{k}$

Defn: Let C be a smooth curve given by $\vec{r}(t)$ defined on the close interval $[a,b]$. For $a \le t \le b$, the arc length function is given by

$$s(t) = \int_a^t \|\vec{r}'(u)\|\,du$$

The arc length s is called the arc length parameter.

By the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = \|\vec{r}\,'(t)\|.$$

which is never zero since C is smooth. If $s = s(t)$ the arc length function, has an inverse, then we write the inverse $t = t(s)$ and, furthermore, we may re-parametrize C by s : $\vec{r}(t(s))$ = the composition of $\vec{r}(t)$ and $t = t(s)$

$$\vec{r}(t(s)) = \vec{r}(s)$$

$$\left\|\frac{d}{ds}\vec{r}(s)\right\| = \left\|\frac{d}{dt}\vec{r}(t)\cdot\frac{dt}{ds}\right\| = \frac{1}{\frac{ds}{dt}}\|\vec{r}\,'(t)\| = 1$$

<u>Theorem 12.7</u>: If C is a smooth curve given by

$\vec{r}(s) = x(s)\,\vec{i} + y(s)\,\vec{j}$ or $\vec{r}(s) = x(s)\,\vec{i} + y(s)\,\vec{j} + z(s)\,\vec{k}$

where s is the arc length parameter, then $\|\vec{r}\,'(s)\| = 1$

$'$ denotes $\frac{d}{ds}$

Moreover, if t is any parameter for the vector-valued function $\vec{r}$ such that $\|\vec{r}\,'(t)\| = 1$, then t must be the arc length parameter ← This is incorrect!

for example, $t = s+1$.

<u>Def'n</u>: Let C be a smooth curve (in the plane or in the space) given by $\vec{r}(s)$, where s is the arc length parameter. The curvature K at s is given by

$$K = \left\|\frac{d\vec{T}}{ds}\right\| = \|T'(s)\|.$$

<u>Example</u>: Show that the curvature of a circle of radius r is $K = \frac{1}{r}$.

<u>Remark</u>: The circle of curvature, the radius of curvature at a point on a curve.

Theorem 12.8: If C is a smooth curve given by $\vec{r}(t)$, then the curvature K of C at t is given by

$$K = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}.$$

Theorem 12.9: If C is the graph of a twice-differentiable function given by $y=f(x)$, then the curvature K at the point (x,y) is given by $K = \dfrac{|y''|}{\left[\sqrt{1+(y')^2}\right]^3}$.

pf: Parametrize C by $\vec{r}(x) = x\vec{i} + f(x)\vec{j}$.

Can be omitted → Theorem 12.10: If $\vec{r}(t)$ is the position vector for a smooth curve C, then the acceleration vector is given by

$$\vec{a}(t) = \frac{d^2s}{dt^2}\vec{T} + K\left(\frac{ds}{dt}\right)^2\vec{N}$$

where K is the curvature of C and $\frac{ds}{dt}$ is the speed.

pf: Recall

$$\begin{aligned} \vec{a}(t) &= a_T\vec{T} + a_N\vec{N} \\ &= D_t(\|\vec{v}\|)\vec{T} + \|\vec{v}\|\|\vec{T}'\|\vec{N} \\ &= \frac{ds^2}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2 K\vec{N}. \end{aligned}$$

$\|\vec{v}\| = \frac{ds}{dt}$

Theorem 12.8