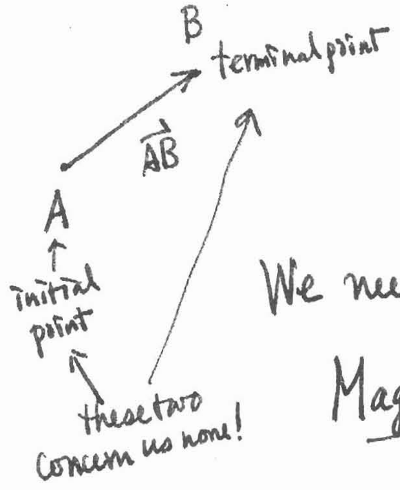
Chapter 11: Vectors and the Geometry of Space.

"Magnitude" — not sufficient to describe what we experience every day. For example,



→ 5 steps, turn back and shoot.
forward.

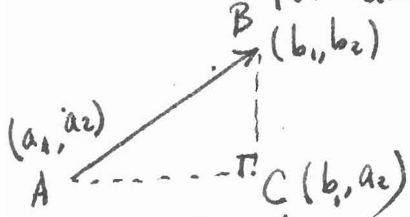
We need "direction"!

Magnitude + Direction ⇝ Vectors (or called, directed line segments)

After introducing coordinates (for points in space or on plane) we have component form of vectors.

For example, if $A=(a_1,a_2)$, $B=(b_1,b_2)$, then

$$\overrightarrow{AB} = (b_1-a_1,\ b_2-a_2)$$



$\overrightarrow{AB}=\overrightarrow{AC}+\overrightarrow{CB}$

(3-D vectors' coordinate form is similarly defined)

$$\|\overrightarrow{AB}\| = \sqrt{(b_1-a_1)^2+(b_2-a_2)^2}. \quad \text{(3D is similar)}.$$

length, magnitude, norm

If $\|\overrightarrow{AB}\|=1$, then $\overrightarrow{AB}$ is called a unit vector.

Let $\vec{u}=(u_1,u_2)$ and $\vec{v}=(v_1,v_2)$, c be a number (or called a scalar)

$\vec{u}+\vec{v} = (u_1+v_1,\ u_2+v_2)$ (space vectors' addition and scalar multiplication are similarly defined)

$c\vec{u} = (cu_1, cu_2)$

$\vec{0}=(0,0)$

$\vec{u}=\vec{v}$ if $\vec{u}-\vec{v} := \vec{u}+(-\vec{v}) = \vec{0}$.

$\Rightarrow$ $\vec{u}+\vec{v}=\vec{v}+\vec{u}$, $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$, $\vec{u}+\vec{0}=\vec{u}$

$\vec{u}+(-\vec{u})=\vec{0}$, $c(d\vec{u})=(cd)\vec{u}$, $(c+d)\vec{u}=c\vec{u}+d\vec{u}$

$c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$, $1\vec{u}=\vec{u}$, $0\vec{u}=\vec{0}$, $\|c\vec{u}\|=|c|\,\|\vec{u}\|$

Standard unit vectors, $\vec{i}:=(1,0)$, $\vec{j}:=(0,1)$ (in plane)

$\vec{i}:=(1,0,0)$, $\vec{j}:=(0,1,0)$, $\vec{k}:=(0,0,1)$

Def'n: Two nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel if there exists a scalar $c \neq 0$ so that $\vec{v} = c\vec{u}$.

(such a $c > 0$ $(<)$, we say that $\vec{u}$ and $\vec{v}$ point to the same direction (opposite).)

Dot Product (or Inner Product, Scalar Product)

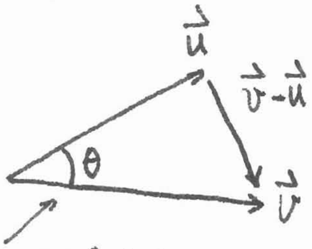
Let $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$, then $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$.

(3D: $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$, if $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$.)

Easy to check: (1) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$, (2) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

(3) $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$, (4) $\vec{0} \cdot \vec{v} = 0$, (5) $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

Let $\vec{u}$ and $\vec{v}$ be two (nonzero) vectors



when $0 \le \theta \le \pi$, it is called the angle between $\vec{u}$ and $\vec{v}$.

By the law of cosine:

$$\|\vec{v} - \vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta.$$

$$\Rightarrow 2\|\vec{u}\|\|\vec{v}\|\cos\theta = \|\vec{u}\|^2 + \|\vec{v}\|^2 - \underbrace{\|\vec{v} - \vec{u}\|^2}_{\substack{\| \\ (\vec{v}-\vec{u}) \cdot (\vec{v}-\vec{u}) \\ = \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{u}\|^2}}$$

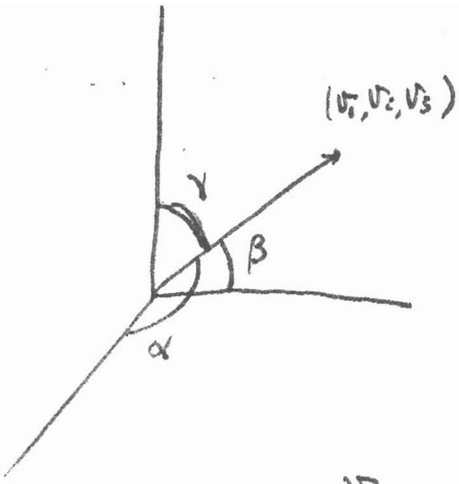
$$= 2\,\vec{u} \cdot \vec{v}$$

$$\Rightarrow \cancel{we}\ \cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}.$$

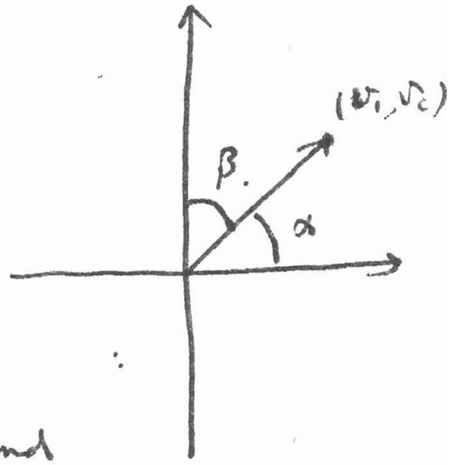
Def'n: The vectors $\vec{u}$ and $\vec{v}$ are ~~orthogonal~~ orthogonal if $\vec{u} \cdot \vec{v} = 0$ } Notation: $\vec{u} \perp \vec{v}$

or say $\vec{u}$ is perpendicular to $\vec{v}$.

Remark: The zero vector and any vector ~~can~~ ~~be~~ are always orthogonal! ~~to any vectors~~!



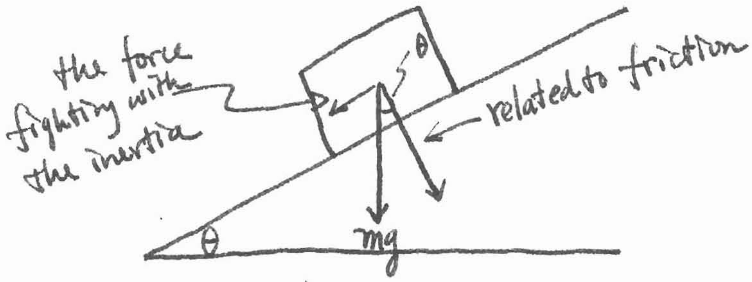
or more easily



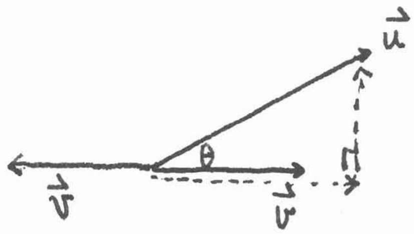
$\cos\alpha = \frac{v_1}{\|v\|}$, $\cos\beta = \frac{v_2}{\|v\|}$ and

$\cos\gamma = \frac{v_3}{\|v\|}$ — called, direction cosines.

Back to high school physics:



Let $\vec{v}$ be a nonzero vector.



Given a vector $\vec{u}$,

$$\left(\|\vec{u}\|\cos\theta\right)\cdot\frac{\vec{v}}{\|\vec{v}\|}$$

$\because \vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$ $\longrightarrow$ $= \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}\cdot\frac{\vec{v}}{\|\vec{v}\|}$

$\therefore$ let $\vec{w}_1 := \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\,\vec{v}$ $\leftarrow$ parallel to $\vec{v}$

and let $\vec{w}_2 := \vec{u} - \vec{w}_1$

$\Rightarrow$ $\vec{u} = \vec{w}_1 + \vec{w}_2$ and $\vec{v}\cdot\vec{w}_2 = \vec{u}\cdot\vec{v} - \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\cancel{\vec{v}\cdot\vec{v}}$

i.e., $\vec{w}_2 \perp \vec{v}$ (of course $\overset{=0}{\vec{w}_2 \perp \vec{w}_1}$)

$\vec{w}_1$: called the projection of $\vec{u}$ onto $\vec{v}$, or the vector component of $\vec{u}$ along $\vec{v}$; <u>notation</u>: $\vec{w}_1 = \mathrm{proj}_{\vec{v}}\,\vec{u}$.

$\vec{W}_2$: Called the vector component of $\vec{u}$ orthogonal to $\vec{v}$.

Cross Product (or Outer Product) — only valid for 3D vectors.

Let $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ be two vectors in space. The cross product of $\vec{u}$ and $\vec{v}$ is the vector

$$\vec{u}\times\vec{v} = (u_2v_3 - u_3v_2)\vec{i} + (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}.$$

Using the language of expanding along a row while ~~calculating~~ computing determinant of matrices, we can reformulate $\vec{u}\times\vec{v}$ as follows:

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \overset{\substack{\text{expanding} \\ \text{along the 1st} \\ \text{row}}}{=\!=} (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}.$$

This reformulation shall bring much ~~can~~ easier ways to understand properties relating to cross product, for examples, $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$, $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$,

$c(\vec{u}\times\vec{v}) = (c\vec{u})\times\vec{v} = \vec{u}\times(c\vec{v})$, $\vec{u}\times\vec{0} = \vec{0}\times\vec{u} = \vec{0}$, $\vec{u}\times\vec{u} = \vec{0}$

$\vec{u}\cdot(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\cdot\vec{w}$

note

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1)$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

triple product
scalar

$\vec{u}$ $\vec{v}$ $\vec{w}$

Q.: $$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$\vec{u}\cdot(\vec{v}\times\vec{w}) \qquad \vec{w}\cdot(\vec{u}\times\vec{v}) \qquad \vec{v}\cdot(\vec{w}\times\vec{u})$$

Theorem 11.8 Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in space, and let θ be the angle between $\vec{u}$ and $\vec{v}$.

(1) $\vec{u}\times\vec{v} \perp \vec{u}$, $\vec{u}\times\vec{v} \perp \vec{v}$.

(2) $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\theta$.

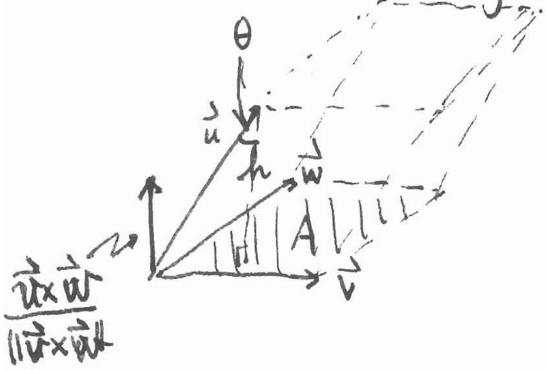
(3) $\vec{u}\times\vec{v} = 0$ iff $\vec{u}$ and $\vec{v}$ are scalar multiples of each other

(4) $\|\vec{u}\times\vec{v}\|$ = are of parallelogram having $\vec{u}$ and $\vec{v}$ as adjacent edges.

Compute $\|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}\cdot\vec{v})^2$. in terms of components of vectors.

Theorem 11.10 Geometric Property of Triple Scalar Product.

The volume V of a parallelepiped with vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ as adjacent edges is given by $V = |\vec{u}\cdot(\vec{v}\times\vec{w})|$

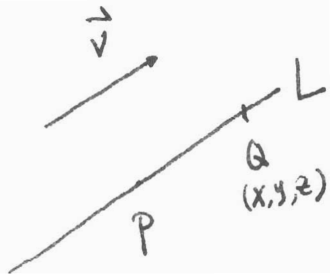


$$\begin{aligned} V &= Ah \\ &= \|\vec{v}\times\vec{w}\|\,\big|\,\|\vec{u}\|\cos\theta\,\big| \\ &= \|\vec{v}\times\vec{w}\|\,\left|\,\vec{u}\cdot\frac{(\vec{v}\times\vec{w})}{\|\vec{v}\times\vec{w}\|}\right| \\ &= |\vec{u}\cdot(\vec{v}\times\vec{w})|. \end{aligned}$$

§ 11.5 : Lines and Planes in Space.

Key: Find (or compute) ~~imper~~ crucial data, and sketch if necessary.

Given a vector $\vec{v} = (a, b, c)$ and a point $P = (x_1, y_1, z_1)$, a line L ~~paral~~ parallel to $\vec{v}$ and passing through P is



$\overrightarrow{PQ} \,/\!/\, \vec{v}$

$\uparrow$ since $L \,/\!/\, \vec{v}$

$\equiv$ there exists a number, say t, so that $\overrightarrow{PQ} = t\vec{v}$

$$\Leftrightarrow (x-x_1, y-y_1, z-z_1) = t(a,b,c)$$

$$\therefore \quad x = x_1 + ta, \; y = y_1 + tb, \; z = z_1 + tc.$$

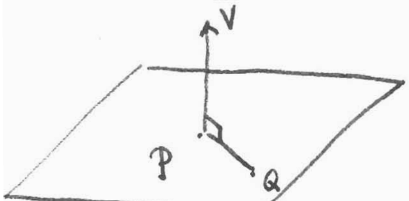
— parametric form

symmetric form.

(may also be written as $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c}$ if none of a, b, c is zero)

Example: Find a set of parametric equations of the line that passes through the points $(-2,1,0)$ and $(1,3,5)$.

Once again, let $\vec{v} = (a,b,c)$ and $P = (x_1, y_1, z_1)$ be given. A plane normal to $\vec{v}$ and passing through P is



$$\overrightarrow{PQ} \cdot \vec{v} = 0.$$

$$(x-x_1, y-y_1, z-z_1) \cdot (a,b,c) = 0$$

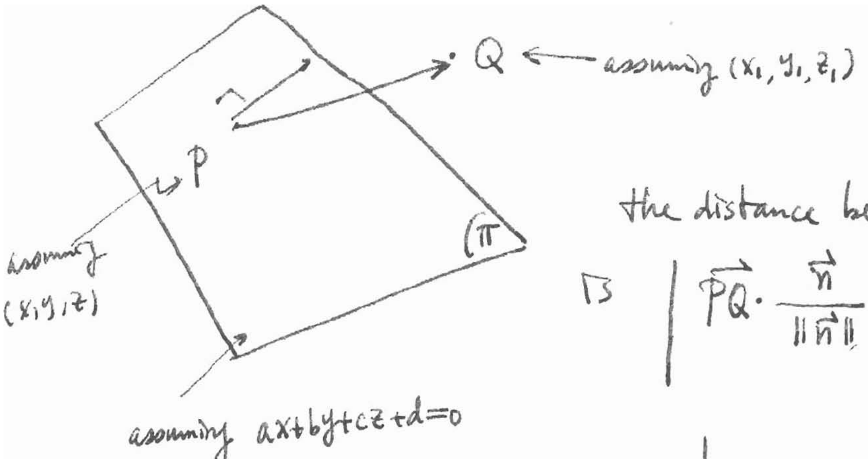
$$\Leftrightarrow a(x-x_1) + b(y-y_1) + c(z-z_1) = 0.$$

— standard form.

or $\quad ax + by + cz + d = 0$ — general form.

Example: Find the general equation of the plane containing the points $(2,1,1)$, $(0,4,1)$, and $(-2,1,4)$.

Example: Find the angle between the two planes given by $x - 2y + z = 0$, $2x + 3y - 2z = 0$, and find parametric equations of their line of intersection.



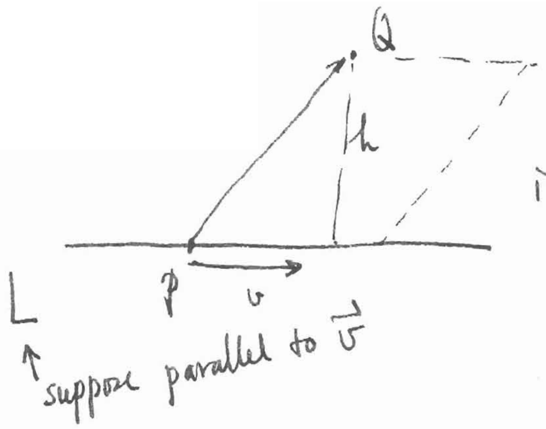
the distance between Q and the plane Π

is $\left| \vec{PQ} \cdot \dfrac{\vec{n}}{\|\vec{n}\|} \right|$ where $\vec{n}$ is normal to Π

$$= \left| (x_1 - x, y_1 - y_0, z_1 - z) \cdot \frac{(a, b, c)}{\sqrt{a^2+b^2+c^2}} \right|$$

$$= \frac{1}{\sqrt{a^2+b^2+c^2}} \left| a(x_1 - x) + b(y_1 - y_0) + c(z_1 - z_0) \right|$$

$$= \frac{1}{\sqrt{a^2+b^2+c^2}} \left| ax_1 + by_1 + cz_1 + d \right|$$



the distance between Q and the line L

is

$$\|\vec{v}\| h = \| \vec{PQ} \times \vec{v} \|$$

$$\therefore h = \left\| \vec{PQ} \times \frac{\vec{v}}{\|\vec{v}\|} \right\|$$

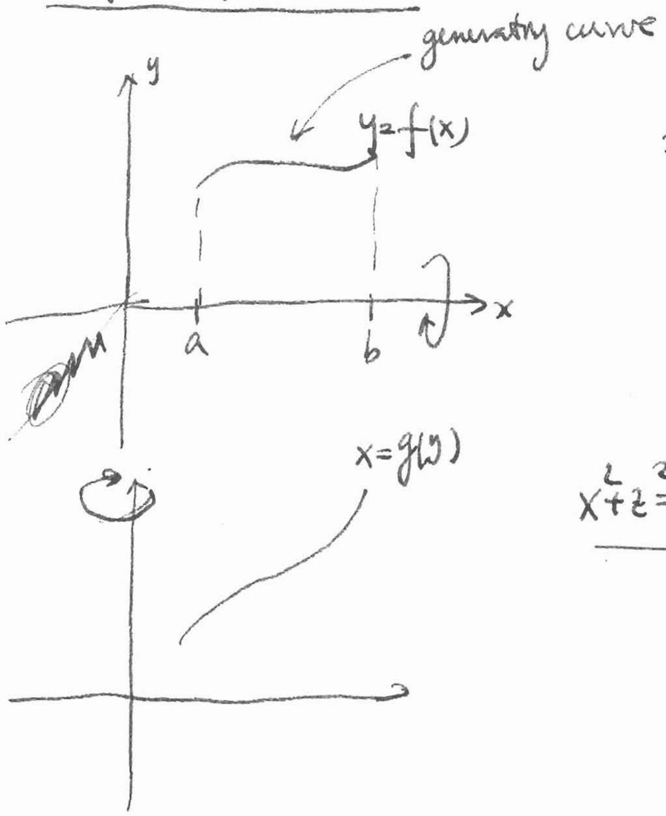
Remark: This sol'n is valid in 3D.

§11.6 Surfaces in Space.

- cylindrical surfaces — or simply, cylinder
- quadratic surfaces
- surfaces of revolution

Def'n: Let C be a curve in a plane and let L be a line not in a parallel plane. The set of all lines parallel to L and intersecting C is called a cylinder. C is called the generating curve (or directrix) of the cylinder, and the parallel lines are called rulings.

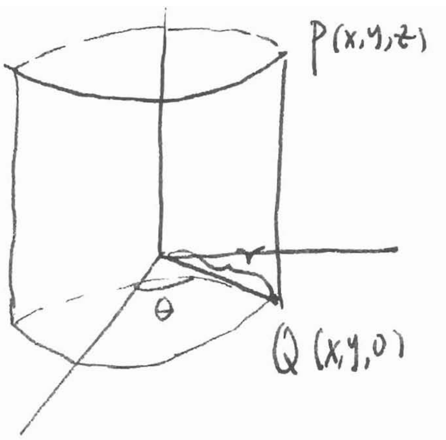
¶ Surfaces of revolution:



$$z^2+y^2=f^2(x)$$
$$a\le x\le b$$

$$x^2+z^2=g^2(y)$$

§11.7. Cylindrical Coordinates, and Spherical Coordinates.



$$r=\sqrt{x^2+y^2}$$
$$\tan\theta=\frac{y}{x}$$

polar coordinates on the x-y plane

+

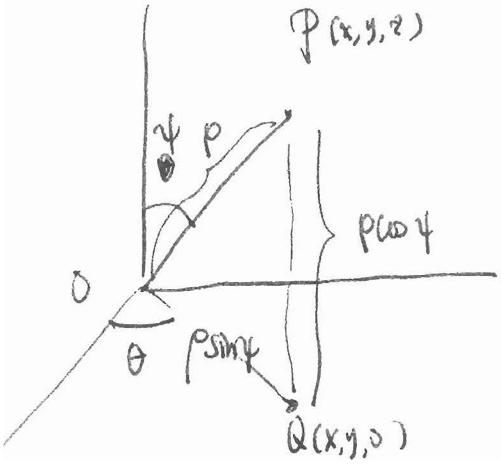
the z-coordinate.

$x=r\cos\theta,\ y=r\sin\theta,\ z=z.$

Since it involves the polar coordinates on $\mathbb{R}^2$, whatever we should be careful then, we should be careful now, ~~for example, the~~

cylindrical coordinates
(r,θ,z)

Spherical Coordinates



$$\rho = \sqrt{x^2+y^2+z^2}$$

$$\cos\psi = \frac{z}{\rho}.$$

$$\tan\theta = \frac{y}{x}$$

$$x = \rho\sin\psi\cos\theta$$

$$y = \rho\sin\psi\sin\theta$$

$$z = \rho\cos\psi.$$