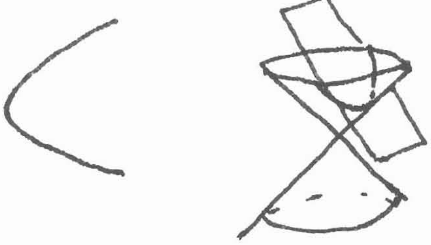
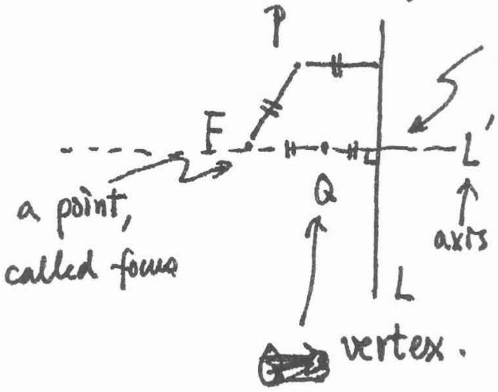
Chapter 10 Conics, Parametric Equations, and Polar Coordinates.

§10.1 Conics and Calculus.

Parabola: ① As a conic section



② From analytic geometry.



$\{ P : \overline{PF} = \text{dist}(P, L) \}$

this set forms a parabola

③ Equation (Standard Form)

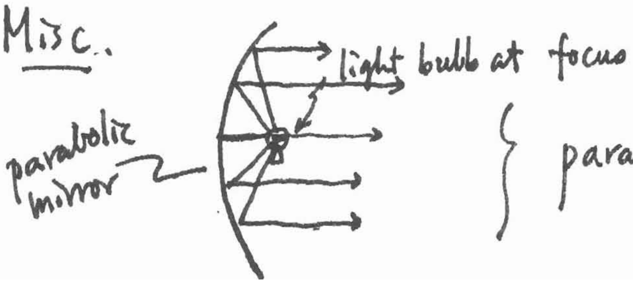
(i) vertex (h,k), directrix $y = k-p$.

$\Rightarrow$ the standard form is $(x-h)^2 = 4p(y-k)$ (with vertical axis) and focus $(h, k+p)$

(ii) vertex (h,k), directrix $x = h-p$

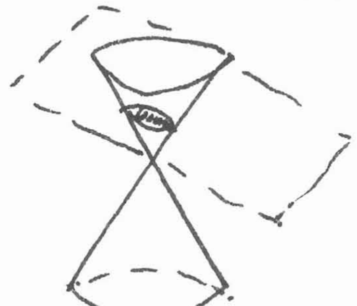
$\Rightarrow$ the standard form is $(y-k)^2 = 4p(x-h)$ (with horizontal axis) and focus $(h+p, k)$

(4) Misc.



parallel light.

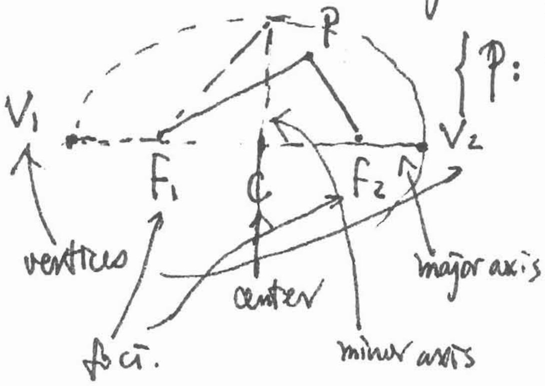
Ellipse: ① As a conic section



② From analytic geometry

$a-c+2c+(a-c)=2a$

|||



$\{P : \overline{PF_1}+\overline{PF_2}=\text{constant}\}$

this set is an ellipse.

③ Equation (standard form)

center (h,k), the length of major axis $2a$ } $a>b$

—— // —— minor axis $2b$

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad \longleftarrow \text{major axis is horizontal}$$

or

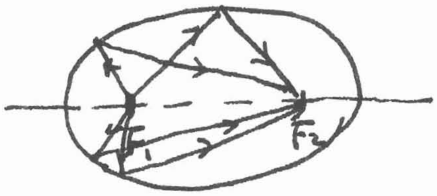
$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1 \quad \longleftarrow \text{major axis is vertical}$$

(the distance between foci is $2c$)

$\Rightarrow c^2=a^2-b^2 \longleftarrow 2\sqrt{b^2+c^2}=2a$

④ Misc.

(i)

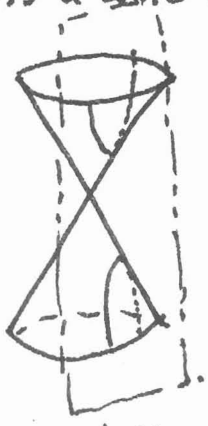


(ii) The eccentricity e of an ellipse is given by the ratio $e=\frac{c}{a}$.

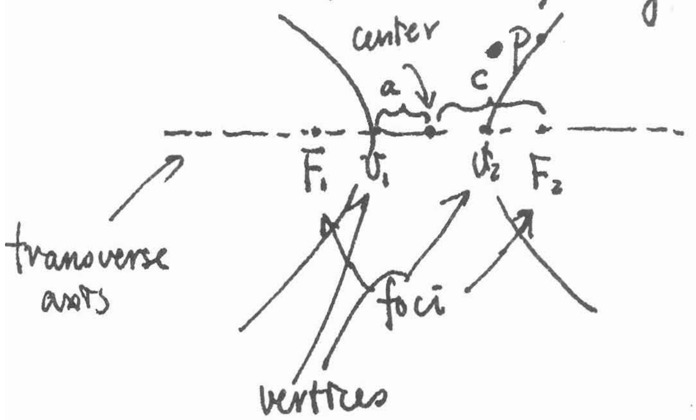
(iii) DO NOT EVEN TRY to calculate the circumference of an ellipse ~~by means~~ — the calculation involves elliptic integral.

Hyperbola

(1) As a conic section

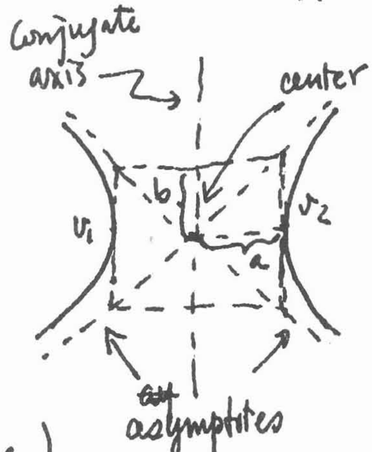


(2) From analytic geometry.



$$\{P : |\overline{PF_1} - \overline{PF_2}| = \overset{2a}{\overset{\shortparallel}{\text{const.}}}\}$$

this set is a hyperbola



$$b^2 = c^2 - a^2$$

(3) Equation (Standard Form)

The standard form of the equation of a hyperbola with center (h,k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{—— the transverse axis is horizontal}$$

or

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1 \quad \text{—— the tranverse axis is vertical}$$

Moreover, in the former case, the equations of the asymptotes are

$$\frac{x-h}{a} = \pm \frac{y-k}{b} \quad \left(\text{or equivalently, } y = k \pm \frac{b}{a}(x-h)\right),$$

in the latter case, they are

$$\frac{y-k}{a} = \pm \frac{x-h}{b} \quad \left(\text{or equivalently, } y = k \pm \frac{a}{b}(x-h)\right).$$

§10.2: Plane Curves and Parametric Equations

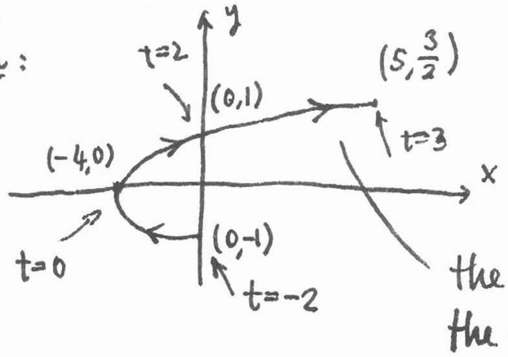
Defn: If f and g are continuous functions of t on an interval I, then the equations $x = f(t)$ and $y = g(t)$ are called parametric equations and t is called the parameter. The set of points (x, y) obtained as t varies over I is called the graph of the parametric equations. Taken together, the parametric equations and the graph are called a plane curve, denoted by C.

Remark: Whenever we mention a plane curve, it usually ~~refers~~ refers to parametric equations (or sometimes called defining equations) rather than the graph.

Example: Sketch the curve described by the parametric equations

$$x = t^2 - 4, \quad y = \frac{t}{2} \quad \text{for } -2 \le t \le 3$$

Soln:



the arrow on the graph indicates the order of increasing values of t — this is called the orientation of the curve

Remark: A different curve has the same graph (and orientation)

$$x = 4t^2 - 4, \quad y = t \quad \text{for } -1 \le t \le \frac{3}{2}.$$

Example: Sketch the curve represented by the equations

$$x = \frac{1}{\sqrt{t+1}} \quad \text{and} \quad y = \frac{t}{t+1}, \quad t > -1$$

by eliminating the parameter and adjusting the domain of the resulting rectangular equation.

Example: Find a set of parametric equations, whose parameter is the slope at the point (x, y), to represent the graph of $y = 1 - x^2$.

Def'n: A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called smooth if f' and g' are continuous on I and not simultaneously 0, except possibly at the endpoints of I. The curve C is called piecewise smooth if it is smooth on each subinterval of some partition of I.

Example: cycloid. $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $a > 0$ for $\theta \geq 0$ is a piecewise smooth curve.

read Example 5 on p. 714.

§10.3 Parametric Equations and Calculus

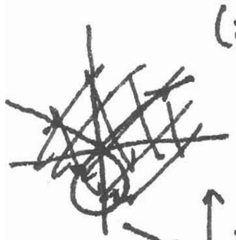
Theorem 10.7 If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at (x, y) is

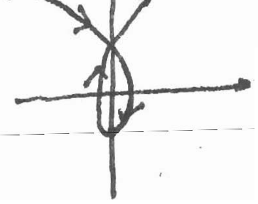
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \text{provided } \frac{dx}{dt} \neq 0.$$

(Sketch) This is merely an application of the Chain Rule.

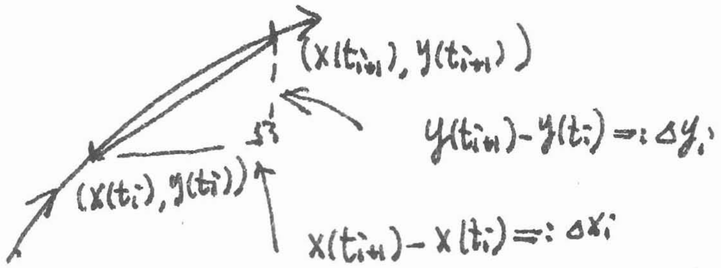
Examples (1) For the curve given by $x = \sqrt{t}$ and $y = \frac{1}{4}(t^2 - 4)$, $t \geq 0$, find the slope and concavity at the point $(2, 3)$.

(2) The prolate cycloid given by $x = 2t - \pi\sin t$ and $y = 2 - \pi\cos t$ crosses itself at the point $(0, 2)$. Find the equations of both tangent lines at this point.





~~Recall: the formula of finding arc length.~~



$$\frac{y(t_{i+1})-y(t_i)}{x(t_{i+1})-x(t_i)} = \frac{y'(\xi_i)}{x'(\xi_i)} \qquad \frac{\Delta y_i}{\Delta x_i} = \frac{y'(\xi_i)}{x'(\xi_i)}$$

$$\Delta L_i = \sqrt{(\Delta x_i)^2+(\Delta y_i)^2} = \sqrt{1+\left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,|\Delta x_i|$$

$$= \sqrt{(x'(\xi_i))^2+(y'(\xi_i))^2}\left|\frac{\Delta x_i}{x'(\xi_i)}\right|$$

$$= \sqrt{(x'(\xi_i))^2+(y'(\xi_i))^2}\left|\frac{x'(\eta_i)}{x'(\xi_i)}\right|\Delta t_i$$

$$t_i < \xi_i, \eta_i < t_{i+1}$$

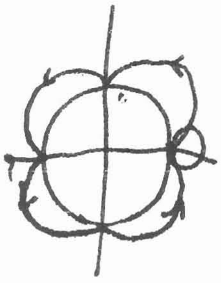
$$\Rightarrow L = \int_a^b \sqrt{(x'(t))^2+(y'(t))^2}\,dt.$$

Remark: For the case of the graph of the curve C is the graph of the function $y=f(x)$, how the formula

$$L = \int_{x_0}^{x_1} \sqrt{1+(f'(x))^2}\,dx$$

relates to

$$L = \int_a^b \sqrt{(x'(t))^2+(y'(t))^2}\,dt\,?$$



Example: A circle of radius 1 rolls around the circumference of a larger circle of radius 4. The epicycloid traced by a point on the circumference of the smaller circle is given by $x=5\cos t-\cos 5t$ and $y=5\sin t-\sin 5t$. Find the distance travelled by the point in one complete trip about the larger circle.

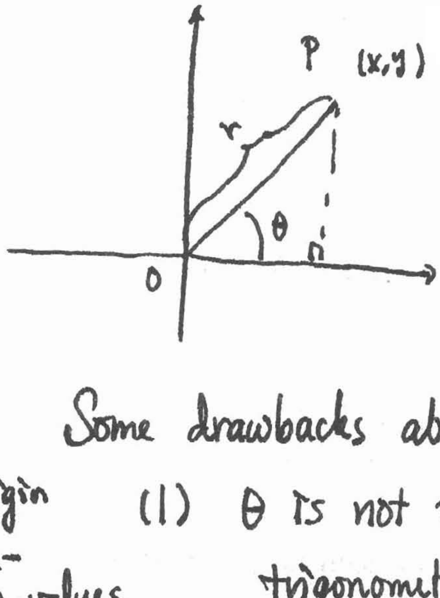
Theorem 10.9: If a smooth curve C given by $x=f(t)$ and $y=g(t)$ does not cross itself on an interval $a\leq t\leq b$, then the area S of the surface of revolution formed by revolving C about the coordinate axes is:

(1) $$S=2\pi\int_a^b g(t)\sqrt{(f'(t))^2+(g'(t))^2}\,dt$$ (revolution about the x-axis; $g(t)\geq 0$)

(2) $$S=2\pi\int_a^b f(t)\sqrt{(f'(t))^2+(g'(t))^2}\,dt$$ (revolution about the y-axis; $f(t)\geq 0$)

Example: Let C be the arc of circle $x^2+y^2=9$ from $(3,0)$ to $\left(\frac{\sqrt{3}}{2},\frac{3\sqrt{3}}{2}\right)$. Find the area of the surface ~~from~~ formed by revolving C about the x-axis

§10.4 Polar Coordinates and Polar Graph.



$x=r\cos\theta$
$y=r\sin\theta$

$r=\sqrt{x^2+y^2}$
~~$\theta=\tan^{-1}\left(\frac{y}{x}\right)$~~ $\tan\theta=\frac{y}{x}$

(r,θ) is called ~~an~~ a polar coordinates of the point P.

Some drawbacks about polar coordinates.

(1) θ is not unique — due to ~~the~~ periodicity of trigonometric functions, $(r,\theta+2n\pi)$, for $n\in\mathbb{Z}$, also represents the same ~~pi~~ point P.

(2) r has to be nonnegative, i.e., $r\geq 0$.

From $x=r\cos\theta$, $y=r\sin\theta$, we have
$$x=(-r)\cos(\theta+\pi),\quad y=(-r)\sin(\theta+\pi)$$
This suggests $(-r,\theta+\pi)=(r,\theta)$. So we extend r from being nonnegative to any real number.

(3) The origin has no well-defined ~~r~~ θ-values. This point is characterized as $r=0$, or say $(0,\theta)$, $\theta\in\mathbb{R}$ represents the origin.

Therefore, strictly speaking, the so-called polar coordinates are not coordinates for points on plane

Polar graph: When we draw the graph of a polar equation, there is no more the origin and the rectangular coordinates system. Instead, we have the pole (playing the role of the origin) and the polar axis

Easy Examples:

(1) $r = 2$. (2) $\theta = \frac{\pi}{3}$. (3) $r = \sec\theta$.

Example: Sketch the graph of $r = 2\cos 3\theta$.

Theorem 10.11: If f is a differentiable function of θ, then the slope of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta},$$

provided the denominator $\neq 0$.

(Sketch) Consider the whole thing in the framework of parametric equations

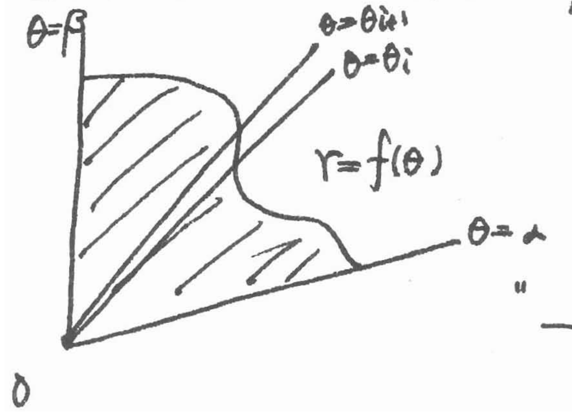
$$x = r\cos\theta = f(\theta)\cos\theta, \quad y = r\sin\theta = f(\theta)\sin\theta.$$

Example: Find the horizontal and verticle tangent lines of $r=\sin\theta$, $0\le\theta\le\pi$.

Theorem 10.12: If $f(\alpha)=0$ and $f'(\alpha)\neq 0$, then the line $\theta=\alpha$ is tangent at the pole to the graph of $r=f(\theta)$.

(Read p. 735 and practice drawing the graphs of the polar equations on that page!)

§10.5 Area and Arc Length in Polar Coordinates.



$$= \frac{1}{2} r^2(\theta_{i+1}-\theta_i)$$
$$= \frac{1}{2} f^2(c_i)\,\Delta\theta_i$$
$$\sum \frac{1}{2} f^2(c_i)\Delta\theta_i \longrightarrow \int_\alpha^\beta \frac{1}{2} f^2(\theta)\,d\theta.$$

Theorem 10.13 If f is continuous and nonnegative on the interval $[\alpha,\beta]$, $0<\beta-\alpha\le 2\pi$, then the area of the region bounded by the graph of $r=f(\theta)$ between the radial lines $\theta=\alpha$ and $\theta=\beta$ is given by

$$A=\frac{1}{2}\int_\alpha^\beta f^2(\theta)\,d\theta$$

Examples (1) Find the area of one petal of the rose-curve given by $r=3\cos 3\theta$

(2) Find the area of the region lying between the inner and outer loops of the limaçon $r=1-2\sin\theta$.

Example Find the area of the region common to the two regions bounded by the curves: $r=-6\cos\theta$ and $r=2-2\cos\theta$

Theorem 10.14: Let f be a function whose derivative is continuous on an interval $\alpha\leq\theta\leq\beta$. The length of the graph of $r=f(\theta)$ from $\theta=\alpha$ to $\theta=\beta$ is

$$L=\int_\alpha^\beta\sqrt{(f(\theta))^2+(f'(\theta))^2}\,d\theta.$$

Example Find the length of the arc from $\theta=0$ to $\theta=2\pi$ for the cardioid $r=f(\theta)=2-2\cos\theta$.

Theorem 10.5: Let f be a function whose derivative is continuous on an interval $\alpha\leq\theta\leq\beta$. The area of the surface formed by revolving the graph of $r=f(\theta)$ from $\theta=\alpha$ to $\theta=\beta$ about the indicated line is

(1) $S=2\pi\int_\alpha^\beta f(\theta)\sin\theta\sqrt{(f(\theta))^2+(f'(\theta))^2}\,d\theta$ (about the polar axis)

(2) $S=2\pi\int_\alpha^\beta f(\theta)\cos\theta\sqrt{(f(\theta))^2+(f'(\theta))^2}\,d\theta$ (about the line $\theta=\frac{\pi}{2}$)

Example: Find the area of the surface formed by revolving the circle $r=f(\theta)=\cos\theta$ about the line $\theta=\frac{\pi}{2}$.