Chapter 8:

§8.1: Review: the known techniques of integration.

Examples: (1) $\int \frac{4}{x^2+9}\,dx$, (2) $\int \frac{4x}{x^2+9}\,dx$, (3) $\int \frac{4x^2}{x^2+9}\,dx$

(4) $\int_0^1 \frac{x+3}{\sqrt{4-x^2}}\,dx$, (5) $\int \frac{x^2}{\sqrt{16-x^6}}\,dx$, (6) $\int \frac{1}{1+e^x}\,dx$

(7) $\int (\cot x)(\ln(\sin x))\,dx$, (8) $\int \tan^2 2x\,dx$.

§8.2: Integration by parts:

From the product rule (or Leibnitz rule)

$$d(uv) = v\,du + u\,dv$$

$$\Rightarrow \quad \int u\,dv = uv - \int v\,du.$$

Examples (1) $\int xe^x\,dx$, (2) $\int x^2 \ln x\,dx$, (3) $\int_0^1 \arcsin x\,dx$.

(4) $\int x^2 \sin x\,dx$, (5) $\int \sec^3 x\,dx$.

$$\int \sec^3 x\,dx = \int \sec x\,d(\tan x)$$

§8.3: Trigonometric Integrals

Consider integrals involving trigonometric functions, like

$$\int \sin^m x \cos^n x\,dx \quad \text{and} \quad \int \tan^m x \sec^n x\,dx.$$

Examples: (1) $\int \sin^3 x \cos^4 x\,dx = \int \sin x(1-\cos^2 x)\cos^4 x\,dx$

(2) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^3 x}{\sqrt{\sin x}}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(1-\sin^2 x)}{\sqrt{\sin x}}\,d(\sin x)$

(3) $\int \cos^4 x\,dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \cdots$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cos^4 x\,dx = \frac{3\pi}{16}$$

Similar integrals are also termed as Wallis's formulas

1. If n is odd $(n \geq 3)$, then
$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{2}{3}\,\frac{4}{5}\,\frac{6}{7}\cdots\frac{n-1}{n}$$

2. If n is even $(n \geq 2)$, then
$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{1}{2}\,\frac{3}{4}\,\frac{5}{6}\cdots\frac{n-1}{n}\,\frac{\pi}{2}.$$

(Try $\int_0^{\frac{\pi}{2}} \sin^n x\,dx$).

Examples: (1) $\displaystyle\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx = \int \frac{(\sec^2 x - 1)}{\sqrt{\sec x}\,(\sec x)}\,d(\sec x)$

(2) $\displaystyle\int \sec^4 3x \tan^3 3x\,dx = \frac{1}{3}\int \sec^3 3x\,(\sec^2 3x - 1)\,d(\sec 3x)$

$$= \frac{1}{3}\int u^5 - u^3\,du = \frac{1}{3}\left(\frac{1}{6}u^6 - \frac{1}{4}u^4\right) + C$$

(3) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^4 x\,dx = \int_0^{\frac{\pi}{4}} \tan^2 x\,(\sec^2 x - 1)\,dx$

$$= \int_0^{\frac{\pi}{4}} \tan^2 x\,\underbrace{\sec^2 x\,dx}_{d(\tan x)} - \int_0^{\frac{\pi}{4}} (\sec^2 x - 1)\,dx.$$

(4) $\displaystyle\int \frac{\sec x}{\tan^2 x}\,dx = \int \frac{\frac{1}{\cos x}}{\frac{\sin^2 x}{\cos^2 x}}\,dx = \int \frac{\cos x}{\sin^2 x}\,dx$

(5) $\displaystyle\int \sin 5x \cos 4x\,dx = \frac{1}{2}\int \sin 9x + \sin x\,dx.$

§8.4 Trigonometric Substitution.
(already introduced!)

Examples. (1) $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$, (2) $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}}$, (3) $\displaystyle\int \frac{dx}{(x^2+1)^{3/2}}$,

(4) $\displaystyle\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$, (5) $\displaystyle\int_{-2}^{-\sqrt{3}} \frac{\sqrt{x^2-3}}{x}\,dx$

§8.5 Partial Fractions.

Like $\frac{1}{6} = \frac{1}{2\times 3} = \frac{1}{2} - \frac{1}{3}$, we wish to write a rational function as a sum of more simpler rational functions.

For example,

$$\frac{1}{x^2-5x+6} = \frac{1}{(x-2)(x-3)} = \frac{-1}{x-2} + \frac{1}{x-3}$$

$$\therefore \int \frac{1}{x^2-5x+6}\,dx = \int \left(\frac{1}{x-3} - \frac{1}{x-2}\right) dx$$

$$= \ln|x-3| - \ln|x-2| + C = \ln\left|\frac{x-3}{x-2}\right| + C.$$

More examples,

Examples: (1) $\displaystyle\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$, (2) $\displaystyle\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}\,dx$

(3) $\displaystyle\int \frac{8x^3+13x}{(x^2+2)^2}\,dx$.

Remarks: (1) $\displaystyle\int \frac{2x^3+x^2-7x+7}{x^2+x-2}\,dx = \int \underbrace{2x-1}_{\text{oper rational function}} + \underbrace{\frac{-2x+5}{x^2+x-2}}_{\text{partial fraction}}\,dx$

the numerator and denominator have common factors! → (2) $\displaystyle\int \frac{x^2-x-2}{x^3-2x-4}\,dx = \int \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}(x^2+2x+2)}\,dx = \int \frac{x+1}{x^2+2x+2}\,dx$

(3) more useful than you ~~the~~ think:

$$\int \frac{\cos x}{\sin x(\sin x - 1)}\,dx \overset{u=\sin x}{=} \int \frac{1}{u(u-1)}\,du.$$

Integration by Tables —— very boring!!

Every one should acquire this "technique" (if it is) without learning! Why teach this in class? Just a waste of time!

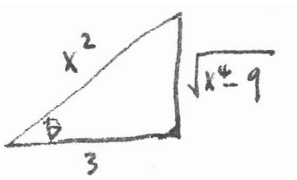
Now, we do examples by hands. Kick the table away!

Examples: (1) $\displaystyle\int \frac{dx}{x\sqrt{x-1}} \overset{\cancel{u=\sqrt{x-1}}}{\underset{u=x-1}{=\!=}} \int \frac{du}{\sqrt{u}\,(u+1)}$

$$\overset{v=\sqrt{u}}{\underset{dv=\frac{du}{2\sqrt{u}}}{=\!=}} \int \frac{2\,dv}{v^2+1} = 2\tan^{-1} v + C$$

$$= 2\tan^{-1}\sqrt{x-1} + C.$$

(2) $\displaystyle\int x\sqrt{x^4-9}\,dx \overset{x^2=3\sec\theta}{\underset{2x\,dx=3\sec\theta\tan\theta\,d\theta}{=\!=}} \int \frac{3}{2}\sec\theta\tan\theta\cdot 3\,|\tan\theta|\,d\theta$



$$= \frac{9}{2}\int \sec\theta\tan^2\theta\,d\theta \overset{\text{Example 5}}{\underset{\text{on p.529}}{=\!=}} \frac{9}{2}\left(\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + C\right)$$

$$\cancel{= \frac{9}{2}\int \sec^3\theta - \sec\theta\,d\theta}$$

$$= \frac{9}{4}\left(\frac{x^2}{3}\frac{\sqrt{x^4-9}}{3} - \cancel{\tfrac{1}{2}}\ln\left|\frac{x^2}{3}+\frac{\sqrt{x^4-9}}{3}\right| + C\right)$$

$$= \frac{1}{4}x^2\sqrt{x^4-9} - \frac{9}{4}\ln\left(x^2+\sqrt{x^4-9}\right) + C$$

(3) $\displaystyle\int \frac{x}{1+e^{-x^2}}\,dx \overset{u=x^2}{\underset{du=2x\,dx}{=\!=}} \frac{1}{2}\int \frac{du}{1+e^{-u}}$

$$= \frac{1}{2}\int 1 - \frac{e^{-u}}{1+e^{-u}}\,du$$

$$= \frac{1}{2}\left(u + \ln(1+e^{-u})\right) + C$$

$$= \frac{1}{2}\left(x^2 + \ln(1+e^{-x^2})\right) + C.$$

(4) $\displaystyle\int x^3 \sin x\,dx \xlongequal[\text{by parts}]{\text{integration}} -x^3\cos x + \int 3x^2\cos x\,dx$

$$= -x^3\cos x + 3x^2\sin x - 3\int 2x\sin x\,dx$$

$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\int \cos x\,dx$$

$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C.$$

(5) $\displaystyle\int \frac{\sqrt{3-5x}}{2x}\,dx \xlongequal{u=3-5x} \int \frac{(\sqrt{u})^2}{u-3}\,\frac{du}{2\sqrt{u}}$

$$\xlongequal{\sqrt{u}=\sqrt{3}\sec\theta} \sqrt{3}\int \frac{\sec^3\theta\tan\theta}{\tan^2\theta}\,d\theta$$

$$\xlongequal{v=\sec\theta} \sqrt{3}\int \frac{v^2}{v^2-1}\,dv$$

$$= \sqrt{3}\int 1+\frac{1}{v^2-1}\,dv$$

$$= \sqrt{3}\left(v+\frac{1}{2}\ln\left|\frac{v-1}{v+1}\right|\right)+C$$

$$= \sqrt{3-5x}+\frac{\sqrt{3}}{2}\ln\left|\frac{\sqrt{3-5x}-\sqrt{3}}{\sqrt{3-5x}+\sqrt{3}}\right|+C.$$

(6) $\displaystyle\int \frac{\sin 2x}{2+\cos x}\,dx = 2\int \frac{\sin x\cos x}{2+\cos x}\,dx$

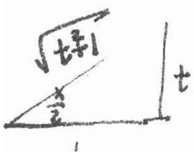
$$\xlongequal{u=\cos x} -2\int \frac{u}{2+u}\,du$$

$$= -2\int 1-\frac{2}{u+2}\,du$$

$$= -2\left(u-2\ln|u+2|\right)+C$$

$$= -2\cos x+4\ln|\cos x+2|+C.$$

or half angle technique $t=\tan\frac{x}{2}$.



$$\Rightarrow \sin\frac{x}{2}=\frac{t}{\sqrt{t^2+1}},\quad \cos\frac{x}{2}=\frac{1}{\sqrt{t^2+1}}$$

$$dt = \sec^2\frac{x}{2}\,\frac{1}{2}\,dx \Rightarrow dx = \frac{2\,dt}{t^2+1}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-t^2}{t^2+1}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2t}{t^2+1}$$

$$\therefore \int \frac{\sin 2x}{2+\cos x}\,dx = \int \frac{2\,\frac{1-t^2}{t^2+1}\,\frac{2t}{t^2+1}}{2+\frac{1-t^2}{t^2+1}}\,\frac{2\,dt}{t^2+1}$$

$$= 8\int \frac{\frac{t(1-t^2)}{(t^2+1)^2}}{\frac{3+t^2}{\cancel{t^2+1}}}\,\frac{dt}{\cancel{t^2+1}}$$

$$= -8\int \frac{t(t^2-1)}{(t^2+3)(t^2+1)^2}\,dt. = \cdots$$

§8.7: Indeterminate Forms and L'Hôpital's Rule.

Theorem 8.3 If f and g are differentiable on an open interval (a,b) and continuous on $[a,b]$ s.t. $g'(x) \neq 0$ for any x in (a,b), then there exists a point $c \in (a,b)$ s.t.

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Pf: Let $h(x) := f(x) - \frac{f(b)-f(a)}{g(b)-g(a)}\,g(x)$. ($g(b) \neq g(a)$, why?)

Theorem 8.4: Let f and g be functions that are differentiable on an open interval (a,b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a,b), except possibly at c itself. If $\lim\limits_{x\to c}\frac{f(x)}{g(x)}$ is the indeterminate form $\frac{0}{0}$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is ∞).

This result also applies if $\lim_{x \to c} \frac{f(x)}{g(x)}$ is one of other indeterminate forms: $\frac{\infty}{\infty}$, $-\frac{\infty}{\infty}$, $0 \cdot \infty$, 1^∞, 0^0, $\infty - \infty$

Examples (1) $\lim_{x \to 0} \frac{e^{2x}-1}{x}$, (2) $\lim_{x \to \infty} \frac{\ln x}{x}$,

(3) $\lim_{x \to -\infty} \frac{x^2}{e^{-x}}$, (4) $\lim_{x \to \infty} e^{-x}\sqrt{x}$, (5) $\underbrace{\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x}_{\text{already done this before}}$

(6) $\lim_{x \to 0^+} (\sin x)^x$, (7) $\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

§8.8: Improper Integrals.

We had ~~...~~ learnt the integrals continuous functions over a finite closed interval $[a,b]$.

How do we deal with integrals of other types, for example, discontinuous functions on $[a,b]$, continuous functions on infinite intervals, and mixed type discontinuous functions on infinite intervals. ??

these all called improper integrals!

Example $\int_1^\infty \frac{dx}{x}$ — continuous function $\frac{1}{x}$ on an infinite interval $[1,\infty)$

vs. (div.)

Examples $\int_0^\infty e^{-x}\,dx$, $\int_0^\infty \frac{1}{x^2+1}\,dx$. (conv.)

Examples: $\int_1^{\infty} (1-x)e^{-x}\,dx$, $\underbrace{\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx}$

(conv.)

still a continuous function on an infinite interval $(-\infty, \infty)$.

Example: $\int_0^1 \frac{dx}{\sqrt[3]{x}}$ — a discontinuous function $\frac{1}{\sqrt[3]{x}}$ on a finite closed interval $[0,1]$.

(conv.)

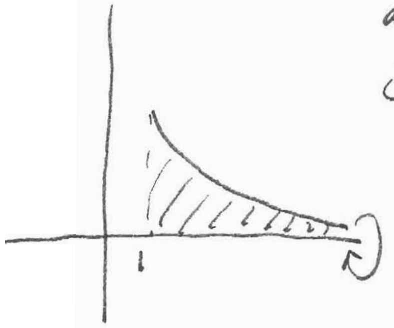
Examples: $\int_0^2 \frac{dx}{x^3}$, $\int_{-1}^2 \frac{dx}{x^3}$ (div.)

Example: $\int_0^{\infty} \frac{dx}{\sqrt{x}\,(x+1)}$ — a discontinuous function $\frac{1}{\sqrt{x}(x+1)}$ on an infinite interval $(0,\infty)$

(conv.) (mixed type)

Theorem 8.5:
$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & \text{if } p>1 \\ \overset{\infty}{\text{div.}} & \text{if } p\le 1. \end{cases}$$

Example: The solid formed by revolving (about the x-axis) the unbounded region lying between the graph of $f(x)=\frac{1}{x}$ and the x-axis $(x\ge 1)$ is called the Gabriel's Horn. Show that this solid has a finite volume but an infinite surface area.



Sol'n: $V = \pi \int_1^{\infty} \frac{1}{x^2}\,dx = \pi.$

$$S = 2\pi \int_1^{\infty} f(x)\sqrt{1+\left[\left(\tfrac{1}{x}\right)'\right]^2}\,dx$$

$$= 2\pi \int_1^{\infty} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$$

div. as $b\to\infty$

$\because \sqrt{1+\frac{1}{x^4}} \ge 1 \quad \therefore \int_1^b \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx \ge \int_1^b \frac{1}{x}\,dx$

$\therefore \int \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$ div.