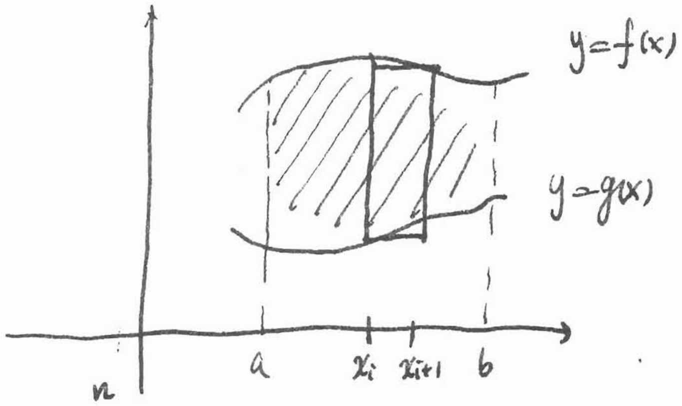
Chapter 7. Applications of Integration.

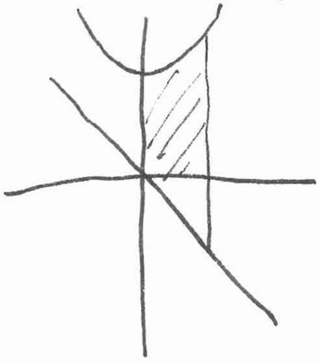
§ 7.1:



$f(x) \geq g(x)$ on $[a,b]$

If $\lim\limits_{\|P\| \to 0} \sum_{i=1}^{\overset{n}{\cancel{\infty}}} (f(c_i) - g(c_i)) \Delta x_i$ exists, the limit is the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$.

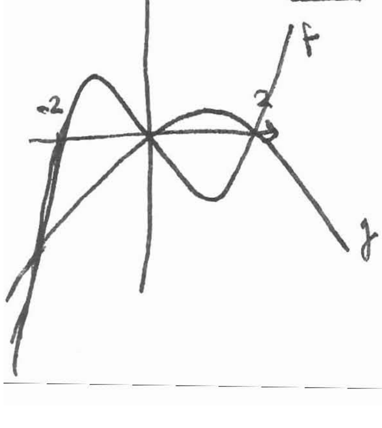
Example: Find the area of the region bounded by the graphs of $y = x^2+2$, $y = -x$, $x=0$, and $x=1$



Soln (Obviously, $x^2+2 > -x$ for $0 \leq x \leq 1$)

$$A = \int_0^1 (x^2+2) - (-x)\,dx = \frac{17}{6}.$$

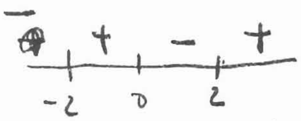
Example: Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$.



Soln: ~~Solve~~ Find the intersection of these two graphs.

$$3x^3 - x^2 - 10x = -x^2 + 2x \Rightarrow \cancel{3}x^3 = \overset{4}{\cancel{12}}x \Rightarrow x(x+2)(x-2) = 0$$

$3x^3 - 12x$



$\therefore$ $f \geq g$ over $[-2, 0]$ and $[2, \infty)$

$$A = \int_{-2}^{0} f(x) - g(x) + \int_0^2 g(x) - f(x) = 24$$

Example: Find the area of the region bounded by the graphs of $x=3-y^2$ and $x=y+1$.

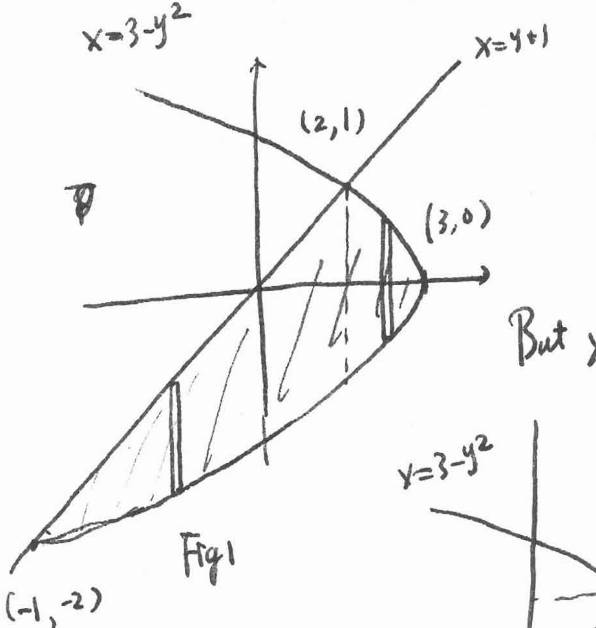


Fig1

$$A=\int_{-1}^{2}(x-1)-(-\sqrt{3-x})\,dx+\int_{2}^{3}\sqrt{3-x}-(-\sqrt{3-x})\,dx$$

$$=\frac{9}{2}$$

But you may also consider the problem like Fig. 2

$$\int_{-2}^{1}(3-y^2)-(y+1)\,dy$$

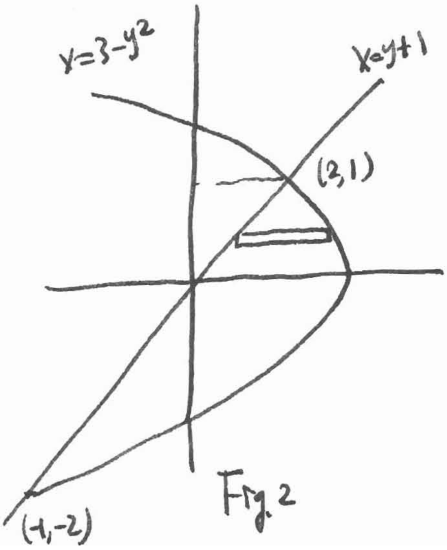


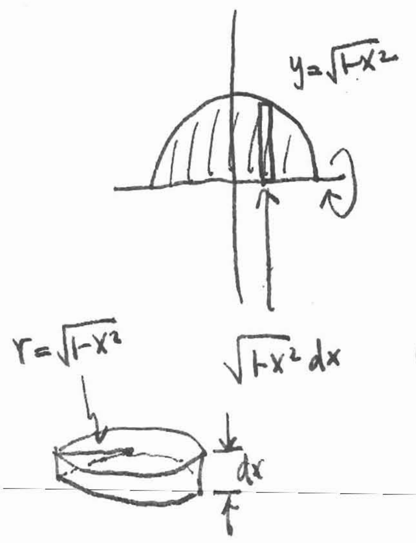
Fig. 2

§ 7.2 Disk method

~~Many~~ Some solids ~~can~~ may be obtained from ~~a~~ planar regions by revolving the regions about some lines, for example,

After revolution, the graph of the function yields a so-called surface of revolution

In the case on the left, the sphere is a surface of revolution. So, how ~~do~~ can we compute the volume of the unit solid ball



$$dV=\pi(\sqrt{1-x^2})^2\,dx$$

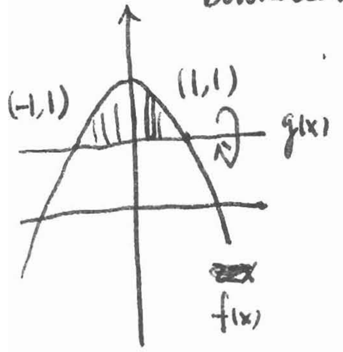
$$\therefore V=\int dV=\int_{-1}^{1}\pi(1-x^2)\,dx=2\pi\int_0^1 1-x^2\,dx$$

$$=2\pi\left(x-\frac{1}{3}x^3\right)\Big|_0^1=\frac{4\pi}{3}.$$

So the ~~whole~~ key point is how to write down dV properly?

$$\pi \cdot (\text{the radius of revolution})^2 \cdot \text{~~(thick~~}(\text{thickness})$$

Example: Find the volume of solid formed by revolving the region bounded by $f(x) = 2-x^2$ and $g(x)=1$ about the line $y=1$



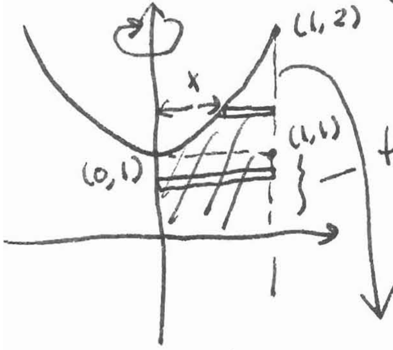
$$V = \int dV = \int_{-1}^{1} \pi \left(f(x)-g(x)\right)^2 dx$$

$$= \int_{-1}^{1} \pi (1-x^2)^2 dx = 2\pi \int_0^1 1-2x^2+x^4 \, dx$$

$$= 2\pi \left(x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right)\Big|_0^1 = \frac{16}{15}\pi.$$

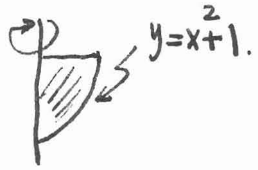
A variant:

Example: Find the volume of the solid formed by revolving the region bounded by the graphs of $y=x^2+1$, $y=0$, $x=0$, and $x=1$ about the y-axis.



this part is revolved to give a cylindrical solid with volume $= \pi \cdot (1)^2 \cdot 1 = \pi =: V_1$

$$V_2 = \int dV = \underbrace{\int_1^2 \pi (1)^2 \cdot dy}_{} - \int_{01}^{2} \pi (x)^2 dy$$



$$= \pi - \pi \int_1^2 (y-1)\, dy = \pi\left(1 - \left(\tfrac{1}{2}y^2 - y\right)\Big|_1^2\right)$$

$$= \pi \frac{1}{2}$$

$$\therefore \; V = V_1 + V_2 = \frac{3}{2}\pi.$$

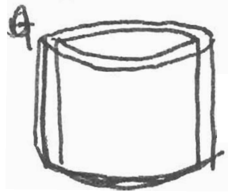
§7.3

We revisit ~~some~~ the preceding examples.

Example: Find the volume of the solid formed by revolving the region bounded by the graphs of $y=x^2+1$, $y=0$, $x=0$, and $x=1$ about the y-axis.

this cylindrical shell has volu dV

this thin rectangular region is revolved to a cylindrical shell: the thickness is dx



$$\pi\cdot(x+dx)^2 f(c_i) - \pi(x)^2 f(c_i)$$

$$= \pi\left(2xdx+(dx)^2\right)f(c_i)$$

$$\sim 2\pi x f(c_i)\,dx$$

$$\therefore V=\int_0^1 2\pi x f(x)\,dx.$$

Shell method

$$= 2\pi\int_0^1 x(x^2+1)\,dx$$

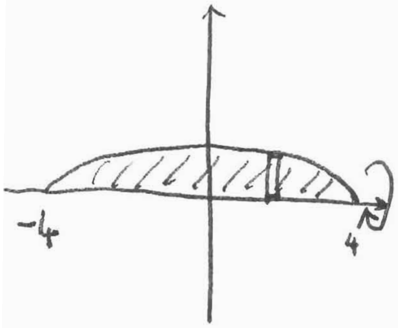
$$= 2\pi\left(\frac{1}{4}x^4+\frac{1}{2}x^2\right)\Big|_0^1 = \frac{6\pi}{4} = \frac{3\pi}{2}.$$

* avoid this in lecture

[**Key point.**

$V=\int dV$ where $dV = 2\pi\cdot$ (radius of ~~ad~~ revolution) × height × thickness]

Example: A pontoon is designed by rotation the graph of $y=1-\frac{x^2}{16}$, $-4\le x\le 4$, about the x-axis. Find the volume of this pontoon.

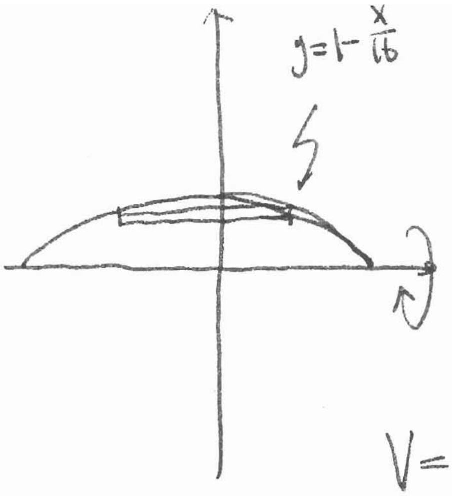


$$V=\int dV = \int_{-4}^{4}\pi\cdot\left(1-\frac{x^2}{16}\right)^2dx.$$

$$= 2\pi\int_0^4 1-\frac{x^2}{8}+\frac{x^4}{256}\,dx$$

$$= \frac{64}{15}\pi.$$

~~Set up~~ Complete the calculation by using the shell method.



~~[illegible]~~

$$dV \sim \pi(y+dy)^2\, 8\sqrt{1-y} - \pi y^2\, 8\sqrt{1-y}$$

$$= 2\pi y\, 8\sqrt{1-y}\, dy$$

$$= 16\pi y\sqrt{1-y}\, dy$$

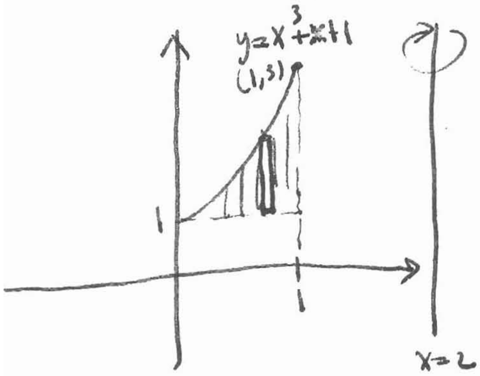
$$V = \int_0^1 16\pi y\sqrt{1-y}\, dy = 16\pi\int_0^1 y\sqrt{1-y}\, dy$$

$$\overset{u=1-y}{\underset{du=-dy}{=}} 16\pi\int_0^{10} (1-u)\sqrt{u}\, du = 16\pi\left(\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right)\Big|_0^1$$

$$= 16\pi\left(\frac{2}{3} - \frac{2}{5}\right) = 32\pi\,\frac{2}{15} = \frac{64}{15}\pi.$$

Example: Find the volume of the solid formed by revolving the region bounded by the graphs of $y=x^3+x+1$, $y=1$, and $x=1$ about the line $x=2$.

$$f(x) = x^3+x+1-1$$



$$dV = \pi(2-x)^2 f(x) - \pi\big(2-(x+dx)\big)^2 f(x)$$

$$= \pi(dx)(4-2x-dx)(x^3+x+1-1)$$

$$= 2\pi(2-x)(x^3+x)\,dx$$

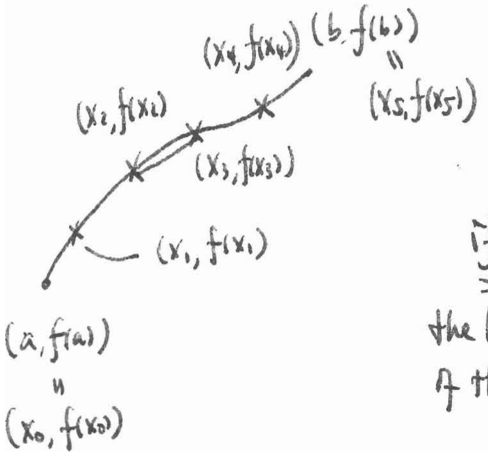
$$V = \int_0^1 2\pi(2-x)(x^3+x)\,dx$$

$$= 2\pi\left(-\frac{1}{5}x^5 + \frac{1}{2}x^4 - \frac{1}{3}x^3 + x^2\right)\Big|_0^1$$

$$= 2\pi\left(-\frac{1}{5} + \frac{1}{2} - \frac{1}{3} + 1\right) = \frac{2\pi}{30}(-6+15-10+30)$$

$$= \frac{29\pi}{15}$$

§ 7.4:



$$\sqrt{(x_{i+1}-x_i)^2+(f(x_{i+1})-f(x_i))^2}$$

$$= \sqrt{1+\left[\frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}\right]^2}\,|\Delta x_i|$$

actuall Δx_i since $\{x_i : i=0,1,\cdots,n\}$ is a partition of $[a,b]$

if f satisfies the hypotheses of the MVT $\qquad \sqrt{1+[f'(c_i)]^2}\,\Delta x_i$

for some $x_i < c_i < x_{i+1}$

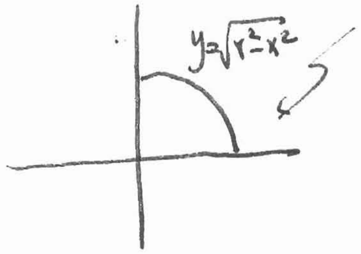
$\therefore$ Suppose $f \in C^1[a,b]$, meaning f has continuous first derivative on $[a,b]$, then

$\sqrt{1+(f')^2}$ is integrable on $[a,b]$

And $\lim_{\|P\|\to 0} \sum_{i=1}^{n} \sqrt{(x_{i+1}-x_i)^2+(f(x_{i+1})-f(x_i))^2}$ exists & equals

$$\int_a^b \sqrt{1+[f'(x)]^2}\,dx.$$

Example:



find the length.

$$\int_0^r \sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$$

$$y' = \frac{-2x}{2\sqrt{r^2-x^2}} = \frac{-x}{\sqrt{r^2-x^2}}$$

$$= \int_0^r \frac{r}{\sqrt{r^2-x^2}}\,dx$$

$$\overset{x=r\sin\theta}{\underset{dx=r\cos\theta d\theta}{=\!=}} \int_0^{\frac{\pi}{2}} \frac{r}{r|\cos\theta|}\, r\cos\theta\, d\theta$$

$$= r\int_0^{\frac{\pi}{2}} \frac{r\cos\theta}{r\cos\theta}\,d\theta = \frac{\pi r}{2}$$

$\therefore$ the circumference of the circle with radius r

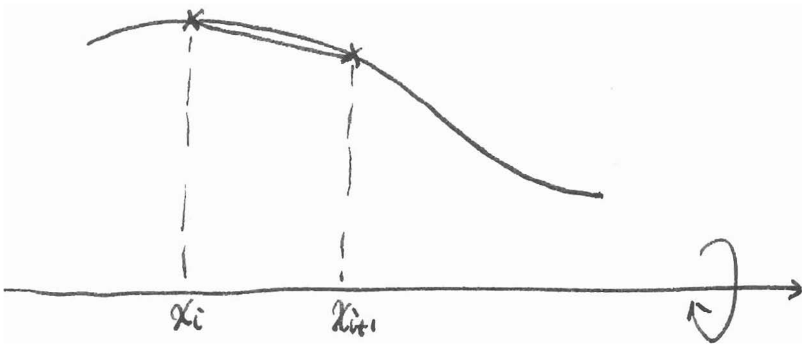
$= 4 \times \frac{\pi r}{2} = 2\pi r.$

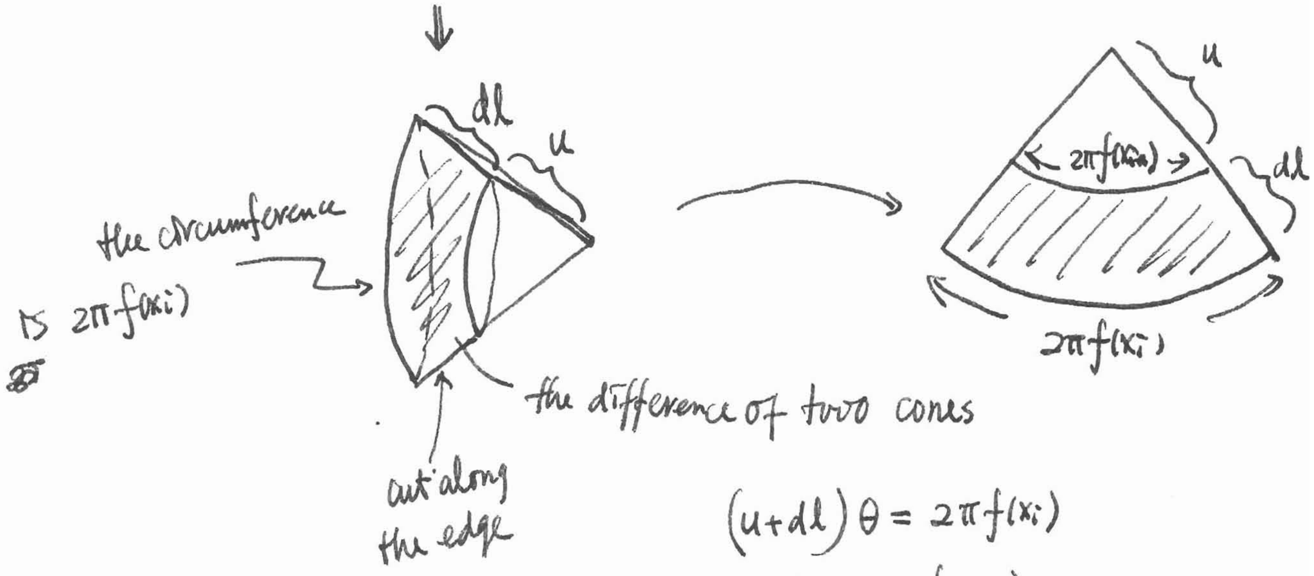
Warning: DO NOT TRY TO FIND THE CIRCUMFERENCE of ANY ELLIPSE!

Example: Find the arc length of the graph of $y=\ln(\cos x)$ from $x=0$ to $x=\frac{\pi}{4}$.

Solu:
$$L=\int_0^{\frac{\pi}{4}}\sqrt{1+\left(\frac{-\sin x}{\cos x}\right)^2}\,dx=\int_0^{\frac{\pi}{4}}\sec x\,dx$$
$$=\ln|\sec x+\tan x|\Big|_0^{\frac{\pi}{4}}=\ln(\sqrt{2}+1).$$

As the graph of a function revolves about some appropriate line, the graph itself gives rise to a surface (of revolution) when the revolution completes. What is the area of the surface of revolution?





$$(u+dl)\theta=2\pi f(x_i)$$
$$u\theta=2\pi f(x_{i+1})$$

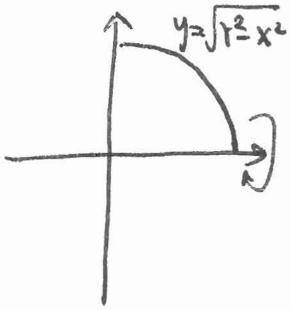
$$\therefore\ dl\cdot\theta = 2\pi\left(f(x_i) - f(x_{i+1})\right)$$

$dA =$  $= \frac{1}{2}(u+dl)^2\theta - \frac{1}{2}u^2\theta = \frac{1}{2}\theta\,(2u+dl)\,dl$

$$\cancel{=} \sim u\theta\, dl = 2\pi f(x_{i+1})\, dl.$$

$\rightarrow$ $dA = 2\pi\cdot$ (radius of rotation) $\cdot\, dl$.

Example:

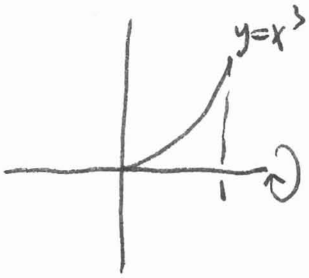


$$dA = 2\pi\sqrt{r^2-x^2}\,\frac{r}{\sqrt{r^2-x^2}}\,dx$$

$$A = \int_0^r 2\pi r\,dx = 2\pi r^2$$

$\Rightarrow$ the surface of the sphere with radius r is $4\pi r^2$.

Example: Find the area of the surface formed by revolving the graph of $f(x) = x^3$ on the interval $[0,1]$ about the x-axis

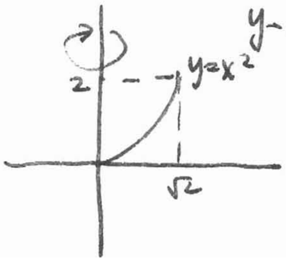


$$A = \int_0^1 2\pi x^3\sqrt{1+(3x^2)^2}\,dx.$$

$$= \frac{\pi}{18}\int_0^1 36x^3\sqrt{1+9x^4}\,dx$$

$$= \frac{\pi}{18}\,\frac{2}{3}\left(1+9x^4\right)^{3/2}\Big|_0^1 = \frac{\pi}{27}\left(\sqrt{1000}-1\right).$$

Example: Find the area of the surface formed by revolving the graph of $f(x)=x^2$ on the interval $[0,\sqrt{2}]$ about the y-axis.



$$A = \int_0^{\sqrt{2}} 2\pi x\sqrt{1+(2x)^2}\,dx = \frac{13}{3}\pi.$$