Chapter 4 Integration.

§4.1

Question: Given a function f, can we find a function whose derivative is f? If we can, how many such functions can be found?

For example, we know that x^2 has derivative $2x$, and functions of the form $x^2 + c$, a ~~box~~ any number, have the same derivative as x^2. Is that all??

Def'n: A function F is an antiderivative of f on an ~~integer~~ interval I if $F'(x) = f(x)$ for all x in I.

Theorem 4.1: If F and G are antiderivatives of f on an interval I, then there is a number C so that $G(x) = F(x) + C$ for all x in I.

called the constant of integration.

This theorem says that $\{F(x) + C : C \text{ is a number}\}$ is the set of all antiderivatives of f on I. ~~This set~~

Notation: $$\underbrace{\int f(x)\,dx}_{\text{realized as the set of all antiderivatives}} = \underbrace{F(x)+C}_{\text{realized as the set describe above}}$$

f is called the integrand, x the variable of integration.

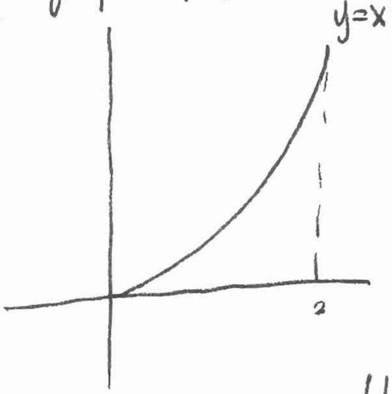
It is ~~always~~ usually difficult to find one antiderivative !!

Example: Find the general solution of $F'(x) = \frac{1}{x^2}$, $x > 0$
and find the particular solution that satisfies ~~the initial condition~~ the condition $F(1) = 0$ ← called the initial condition.

Newton mechanics

§4.2

Question: Find the area of the region bounded by the graph of $f(x) = x^2$ and the x-axis between $x=0$ and $x=2$



(upper sum) $\sum_{k=1}^{n} f\left(\frac{2k}{n}\right)\left(\frac{2}{n}\right)$

$$= 8\sum_{k=1}^{n} \frac{k^2}{n^3} = \frac{8}{n^3}\sum_{k=1}^{n} k^2$$

$$= \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{4}{3}\,\frac{(n+1)(2n+1)}{n^2}$$

(lower sum) $\sum_{k=1}^{n} f\left(\frac{2(k-1)}{n}\right)\frac{2}{n}$

$$= \sum_{k=1}^{n} \frac{8(k-1)^2}{n^3} = \frac{8}{n^3}\sum_{j=0}^{n-1} j^2$$

$$= \frac{8}{n^3}\sum_{j=1}^{n-1} j^2 = \frac{8}{n^3}\,\frac{(n-1)\,n\,(2n-2+1)}{6} = \frac{4}{3}\,\frac{(n-1)(2n-1)}{n^2}$$

§4.3

Riemann ~~gave a general "definition" of finding~~

Def'n: A partition P of the interval $[a,b]$ is a set of finite points $\{x_0, x_1, \dots, x_n\}$ satisfying

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

$$\|P\| := \max_{1 \le i \le n} |x_i - x_{i-1}| \quad \text{or} \quad \Delta x_i = x_i - x_{i-1}$$

~~Def'n: Let P and P' be two partitions of the interval $[a,b]$. P' is finer than P if $P \subset P'$~~

Given a partition P, choose $c_i \in [x_{i-1}, x_i]$ for $1 \le i \le n$, then $\sum_{i=1}^{n} f(c_i)\Delta x_i$ is called a Riemann sum.

Def'n: Let L be a number. If $\forall \epsilon > 0$, $\exists \delta > 0$ so that for every partition with $\|P\| < \delta$ we have

$$\left| L - \sum_{i=1}^{n} f(c_i)\Delta x_i \right| < \epsilon \quad \cancel{\text{each}}$$

for any choice of $\{c_i\}_{i=1}^{n}$, then L is called the limit of the Riemman sum of f on $[a,b]$.

Notation: $L = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$.

Def'n: If f is defined on the interval $[a,b]$, and the limit $\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$ exists, then f is called integrable on $[a,b]$ and the limit is denoted by

$$\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x = \underbrace{\int_a^b f(x)\,dx}_{\text{replaces } L.}$$

The limit is called the definite integral of f from a to b

lower limit upper limit of integration.

Theorem 4.4.: If f is continuous on the interval $[a,b]$, then f is integrable on $[a,b]$.

In fact, for continuous functions on $[a,b]$, it is more conceptually easier ~~way~~ to describe the limit of Riemann sum

On each subinterval $[x_{i-1}, x_i]$ for $i = 1, 2, \cdots, n$, since f is continuous there, $\exists$ $M_i, m_i \in [x_{i-1}, x_i]$ s.t.

$$f(m_i) = \min_{x_{i-1} \leq x \leq x_i} f(x) \quad \text{and} \quad f(M_i) = \max_{x_{i-1} \leq x \leq x_i} f(x)$$

$$\Longrightarrow \quad \sum_{i=1}^{n} f(m_i)\Delta x_i \leq \sum_{i=1}^{n} f(c_i)\Delta x_i \leq \sum_{i=1}^{n} f(M_i)\Delta x_i$$

for any choice of $\{c_i\}_{i=1}^{n}$ where $c_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \cdots, n$.

We call $\underbrace{\sum_{i=1}^{n} f(m_i)\Delta x_i}_{= L(f,P)}$ the lower sum of f w.r.t. the partition P

$\underbrace{\sum_{i=1}^{n} f(M_i)\Delta x_i}_{= U(f,P)}$ the upper sum — " — P

$\|P'\| \leq \|P\|$

$\Uparrow$

We also observe that if $P \subset P'$ (called: P' is finer than P)

then $\quad L(f,P) \leq L(f,P') \leq U(f,P') \leq U(f,P)$

If ~~we do~~ the notation $P \leq P'$ means P' is finer than P,

then $L(f,\cdot)$ is $\nearrow$ and $U(f,\cdot)$ $\searrow$

Moreover, for any partition P, $L(f,P) \leq \underbrace{f(M)}_{= \max_{a \leq x \leq b} f(x)} (b-a) \Big\} U(f,P_0)$ where P_0 is the trivial partition $\{a,b\}$

~~& ∃ sup L(f,P)~~

and

$U(f,P) \geq \underbrace{f(m)}_{= \min_{a \leq x \leq b} f(x)} (b-a) \Big\} L(f,P_0)$

$$L(f,P) \leq L(f,P') \leq \cdots \leq \cdots \leq U(f,P') \leq U(f,P)$$

$\sup L(f,\cdot)$ exists

$\uparrow$

also called the lower integral

$\underline{\int_a^b} f(x)\,dx$

$\inf U(f,\cdot)$ exists

$\uparrow$

also called upper integral

$\overline{\int_a^b} f(x)\,dx \longleftarrow$ if they ~~are~~ equal then f is integrable.

Properties

Def'n: (1) If f is defined at $x=a$, then we define $\int_a^a f(x)dx=0$

(2) If f is integrable on $[a,b]$, then we define $\int_b^a f(x)dx=-\int_a^b f(x)dx$

Theorem 4.6: If f is integrable on the three closed intervals determined by a, b, and c, then $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$.

Theorem 4.7: If f and g are integrable on $[a,b]$, then kf and $f\pm g$ are also integrable on $[a,b]$, where k is a constant, and

(1) $\int_a^b kf(x)\,dx = k\int_a^b f(x)dx$

(2) $\int_a^b \left[f(x)\pm g(x)\right]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$.

Theorem 4.8: (1) If $f \geq 0$ and f is integrable on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$

(2) If f and g are integrable and $f(x)\leq g(x)$ on $[a,b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

§4.4: (The fundamental Theorem of Calculus)

Theorem 4.9: If f is continuous on $[a,b]$ and F is an antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx = F(b)-F(a)$$

※ Not every integrable function has an antiderivative for example:

$$f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$$

on $[-1,1]$ is integrable but has no antiderivative

Pf: Let P be any partition of $[a,b]$

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

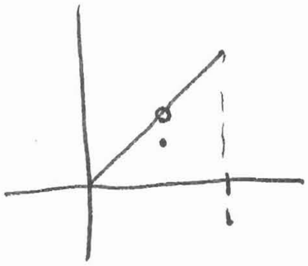
$$F(b)-F(a) = \left(F(x_n)-F(x_{n-1})\right)+\left(F(x_{n-1})-F(x_{n-2})\right)+\cdots + \left(F(x_2)-F(x_1)\right)+\left(F(x_1)-F(x_0)\right)$$

$$=\sum_{i=1}^{n} f'(c_i)\,\Delta x_i \longrightarrow \int_a^b f(x)dx$$

Examples: (1) $\int_0^2 x^2-3\,dx$ (2) $\int_0^2 |2x-1|\,dx$

Theorem 4.10 (Mean Value Theorem for integrals)

If f is continuous on $[a,b]$, then there exists a $c \in [a,b]$ s.t. $\underbrace{\frac{1}{b-a}\int_a^b f(x)\,dx}_{\text{the average of } f \text{ on } [a,b].} = f(c)$



Pf: $\underset{\substack{\min f(x) \\ a\le x\le b}}{f(m)} \le f(x) \le f(M) = \max_{a\le x\le b} f(x)$

$$\Rightarrow f(m)(b-a) \le \int_a^b f(x)\,dx \le f(M)(b-a)$$

$$\Rightarrow f(m) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le f(M)$$

By intermediate value theorem, $\exists\, c$ ~~between~~ $\in [m, M]$ or $[M, m]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

Theorem 4.11 (The second fundamental theorem of Calculus)

If f is continuous on an open interval I containing a, then, $\forall x \in I$, $\underbrace{\int_a^x f(t)\,dt}_{G(x)}$ is differentiable,

and $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$

$\frac{G(x+h)-G(x)}{h}$

$= \frac{1}{h}\int_x^{x+h} f(t)\,dt$

$= \frac{1}{h}\, h\, f(c)$

where $c \in [x, x+h]$ or $[x+h, x]$

$= f(c) \xrightarrow{h\to 0} f(x)$

$\because f$ is conti.

(*) If f has an antiderivative

Pf: Let F be an antiderivative of f on I,

then $\int_a^x f(t)\,dt = F(x) - F(a)$

$$\Rightarrow \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = \frac{d}{dx}F(x) = f(x).$$

Example: Find the derivative of $F(x) = \int_{\pi}^{x^3} \cos t\,dt$

§4.5: Whatever we had for ~~differe~~ derivatives, we have ~~almost~~ "the same" almost for antiderivatives

For example,

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{df}{dx}(x) \pm \frac{dg}{dx}(x) \longleftrightarrow \int f \pm g \, dx = \int f \, dx \pm \int g \, dx.$$

How about chain rule

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x) \longleftrightarrow \text{? change of variables}$$

(or on the text: substitution)

Let's work on a lot of examples.

(1) $\int (x^2+1)(2x)\,dx.$

(2) $\int 5\cos 5x \, dx$

(3) $\int x(x^2+1)^2 dx$

(4) $\int \sqrt{2x-1}\, dx$

(5) $\int x\sqrt{2x-1}\, dx$

(6) $\int \sin^2 3x \cos 3x \, dx$

(7) $\int_0^1 x(x^2+1)^3 dx$

(8) $\int_1^5 \frac{x}{\sqrt{2x-1}}\, dx$

Theorem 4.15: Let f be integrable on $[-a, a]$.

1. If f is even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$
2. If f is odd, $\int_{-a}^{a} f(x)\,dx = 0.$