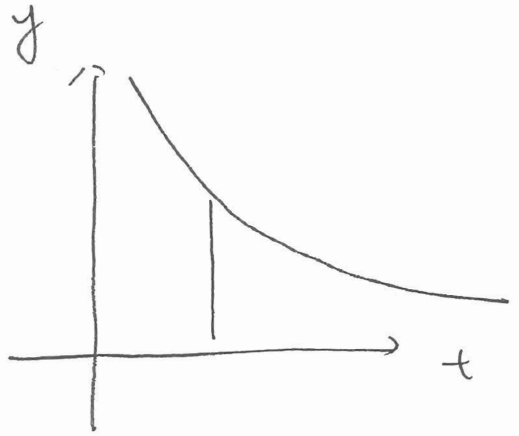
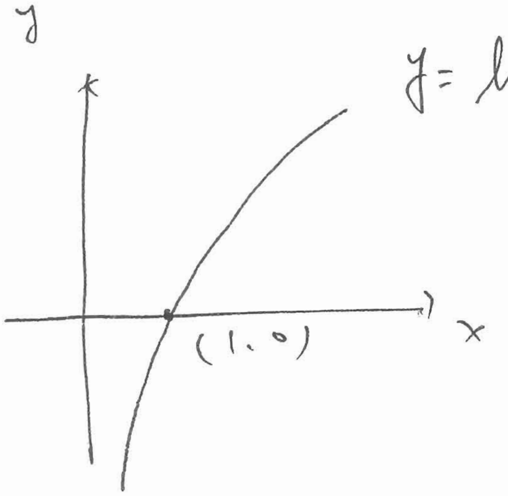
§ 5.1. The Natural Logarithmic function and Differentiation. 11/20

Def: The natural logarithmic function is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$





$\underline{\underline{\ln x}}$: $\begin{cases} \text{domain } (0, \infty) \\ \text{range } (-\infty, \infty) \\ \underline{\text{cont}}, \ \underline{\text{increasing}}, \ \underline{\text{one-to-one}} \\ \text{concave down.} \end{cases}$

Def: The e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t}\,dt = 1$$

$$\boxed{e \approx 2.718281828 46}.$$

Thm: $\begin{cases} \boxed{\dfrac{d}{dx}\ln x = \dfrac{1}{x}, \quad x > 0.} \\ \dfrac{d}{dx}\ln u = \dfrac{u'}{u} \quad u \text{ is a diff function of } x. \end{cases}$

Ex: $\frac{d}{dx}\ln(2x) = \frac{1}{2x}\cdot 2 = \frac{1}{x}$.

$\frac{d}{dx}\ln(x^2+1) = \frac{2x}{x^2+1}$

$\frac{d}{dx}x\ln x = x\cdot\frac{1}{x} + \ln x\cdot 1 = \ln x + 1$

$\cancel{\frac{d}{dx}[(\ln x)^3] =}$

$\frac{d}{dx}(\ln x)^3 = 3(\ln x)^2\cdot\frac{1}{x}$

$\frac{d}{dx}\ln\sqrt{x+1} = \frac{1}{\sqrt{x+1}}\cdot\frac{1}{2}\frac{1}{\sqrt{x+1}} = \frac{1}{2(x+1)}$.

Ex: $\frac{d}{dx}\ln\frac{x(x^2+1)^2}{\sqrt{2x^3-1}}$

$= \frac{d}{dx}[\ln(x(x^2+1)^2) - \ln\sqrt{2x^3-1})$

$= \frac{d}{dx}[\ln x + \ln(x^2+1)^2 - \ln\sqrt{2x^3-1})$

$= \frac{d}{dx}[\ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1))]$

$= \frac{1}{x} + 2\cdot\frac{2x}{x^2+1} - \frac{1}{2}\cdot\frac{1}{2x^3-1}\cdot 6x^2$

$= \frac{1}{x} + \frac{4x}{x^2+1} - \frac{3x^2}{2x^3-1}$

\#

Logarithmic Diff

Ex: Find the derivative of

$$y = \frac{(x-2)^2}{\sqrt{x^2+1}} \quad , \quad x \neq 2 .$$

$$\ln y = \ln \frac{(x-2)^2}{\sqrt{x^2+1}}$$

$$\Rightarrow \ln y = \ln (x-2)^2 - \ln (\sqrt{x^2+1})$$

$$= 2 \ln (x-2) - \frac{1}{2} \ln (x^2+1)$$

$$\therefore \frac{1}{y} \cdot y' = 2 \cdot \frac{1}{x-2} - \frac{1}{2} \cdot \frac{2x}{x^2+1}$$

$$= \frac{2}{x-2} - \frac{x}{x^2+1}$$

$$\therefore y' = \left(\frac{2}{x-2} - \frac{x}{x^2+1} \right) \cdot y$$

$$= \left(\frac{2}{x-2} - \frac{x}{x^2+1} \right) \cdot \left(\frac{(x-2)^2}{\sqrt{x^2+1}} \right).$$

Thm:

$$\begin{cases} \dfrac{d \ln x}{dx} = \dfrac{1}{x} \quad (x > 0), & \dfrac{d \ln|x|}{dx} = \dfrac{1}{x} \\ \dfrac{d \ln u}{dx} = \dfrac{1}{u} \cdot u', & \dfrac{d \ln|u|}{dx} = \dfrac{u'}{u} \end{cases}$$

Ex: Find the derivative of $f(x) = \ln|\cos x|$ (p. 319).

Locate the relative extrema of $y = \ln(x^2+2x+3)$

§. 5.2. The natural Logarithmic function and Integration. $\int \ln x \, dx = ?$

Thm: $\int \frac{1}{x} dx = \ln|x| + C$.

Ex: $\int \frac{2}{x} dx = 2 \int \frac{1}{x} dx$

$= 2 \ln|x| + C$.

Ex: $\int \frac{1}{4x-1} dx \xlongequal[\frac{du}{dx}=4]{u=4x-1} \int \frac{1}{u} \cdot \frac{1}{4} du$

$\int \frac{1}{ax+b} dx = ?$

$= \frac{1}{4} \ln|u| + C$

$= \frac{1}{4} \ln|4x-1| + C$

Ex: (P.322 Ex 3)

Ex: $\int_0^3 \frac{x}{x^2+1}\,dx$

$$\underset{\frac{du}{dx}=2x}{\overset{u=x^2+1}{=\!=}} \int_1^{10} \frac{1}{u}\cdot\frac{1}{2}\,du$$

$$= \frac{1}{2}\ln|u|\Big|_1^{10}$$

$$= \frac{1}{2}[\ln 10 - \ln 1] = \frac{1}{2}\ln 10$$

#.

Ex: $\int \frac{\sec^2 x}{\tan x}\,dx$

$$\underset{\frac{du}{dx}=\sec^2 x}{\overset{u=\tan x}{=}} \int \frac{1}{u}\,du = \ln|u| + c = \ln|\tan x| + C$$

#

Ex: $\int \frac{3x^2+1}{x^3+x}\,dx$

$$\underset{\frac{du}{dx}=3x^2+1}{\overset{u=x^3+x}{=\!=}} \int \frac{1}{u}\,du$$

$$= \ln|x^3+x| + C$$

#.

Ex: $\displaystyle\int \frac{x^2+x+1}{x^2+1}\,dx$

$$= \int \left[1 + \frac{x}{x^2+1} \right] dx$$

$$= \int 1\,dx + \int \frac{x}{x^2+1}\,dx$$

$$= x + \frac{1}{2}\ln(x^2+1) + C$$

#.

Ex: $\displaystyle\int \frac{2x}{(x+1)^2}\,dx$

$$\overset{u=x+1}{=} \int \frac{1}{u^2} \cdot 2(u-1)\,du$$

$$= 2\left[\int \left(\frac{1}{u} - \frac{1}{u^2} \right) du \right]$$

$$= 2\left(\ln|u| + u^{-1} \right) + C$$

$$= 2\left(\ln|x+1| + (x+1)^{-1} \right) + C$$

#.

Ex: Solve the diff. equation $\dfrac{dy}{dx} = \dfrac{1}{x\ln x}$

Sol: $\displaystyle y = \int \frac{1}{x\ln x}\,dx = \int \frac{1}{\ln x}\,d\ln x$

$$= \ln|\ln x| + C$$

#.

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int \frac{1}{\cos x} (-d \cos x)$$

$$= -\ln|\cos x| + C.$$

$$\int \cot x \, dx =$$

⊗ $$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$u = \underline{\underline{\sec x + \tan x}}$

$\frac{du}{dx} = \sec x \tan x + \sec^2 x$

$$\int \frac{1}{u} \, du$$

$$= \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C.$$

#.

$$\int \csc x \, dx =$$

Ex: $$\int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2 x} \, dx$$

§5.3 Inverse functions. 11/20 11/23.

<u>Def</u>: A function g is the inverse of the function f if

$f(g(x)) = x$ for each x in the domain of g

and $g(f(x)) = x$ for each x in the domain of f

$$\begin{cases} x \xrightarrow{g} g(x) \xrightarrow{f} x \\ x \xrightarrow{f} f(x) \xrightarrow{g} x \end{cases}$$

$$\boxed{x \underset{f^{-1}}{\overset{f}{\rightleftarrows}} f(x)}$$

The function g is denoted by f^{-1} ($\underline{f^{-1}(x) \neq \frac{1}{f(x)}}$)

◎ (1) If g is the inverse of f

$\Rightarrow$ f is the inverse of g

(2)

f	f^{-1}
~~domain~~ range $\longleftrightarrow$	~~range~~ domain
domain $\longleftrightarrow$	range.

$$\begin{cases} D(f) = R(f^{-1}) \\ R(f) = D(f^{-1}) \end{cases}$$

(3) f^{-1} is unique.

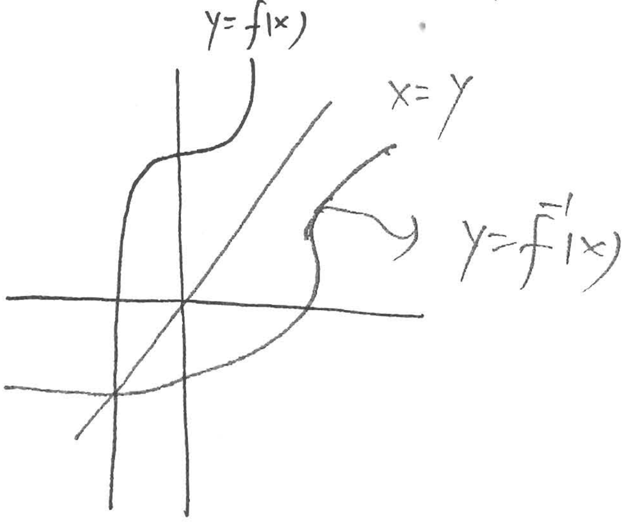
Ex: $\begin{cases} f(x) = x + c \\ \Leftrightarrow \ f^{-1}(x) = x - c. \end{cases}$ $\qquad y = x + c$, $\quad x = y - c$

$\begin{cases} f(x) = cx \\ f^{-1}(x) = \dfrac{x}{c}. \end{cases}$ $\qquad y = cx$, $\quad x = \dfrac{y}{c}$

◎

Thm: The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a)

(f, f^{-1} 對稱於 $x=y$)



Theorem:

(a) A function has an inverse

$\Longleftrightarrow$ it is one-to-one.

(b) if f is strictly monotonic (increasing or decreasing)

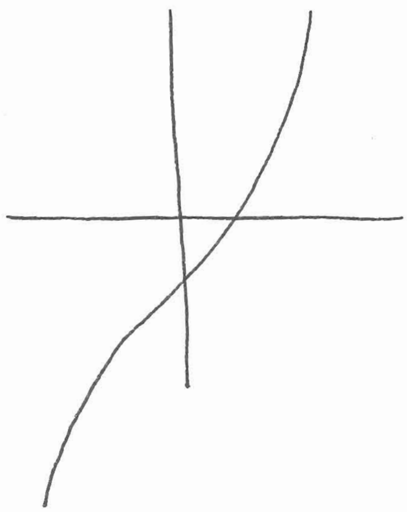
then it is one-to-one and therefore

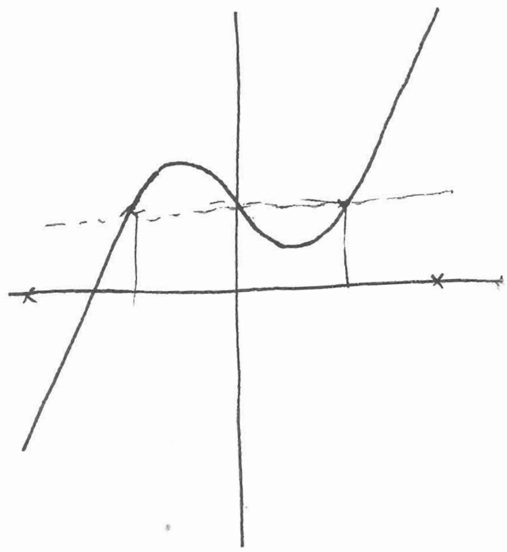
has an inverse.

Ex: Which of the functions has an inverse.

(a) $f(x) = x^3 + x - 1$

(b) $f(x) = x^3 - x + 1$





Ex: Find the inverse of $f(x) = \sqrt{2x-3}$

Sol:

$y = f(x) = \sqrt{2x-3}$ $\quad \boxed{D(f) = \{x ; 2x-3 \geq 0\}}$

$\frac{dy}{dx} = \frac{1}{2}(2x-3)^{-\frac{1}{2}} \cdot 2 = \frac{1}{\sqrt{2x-3}} \geq 0$ on $D(f)$

$\therefore$ y is increasing

$\Rightarrow$ f has an inverse

$y = f(x) = \sqrt{2x-3}$

$y^2 = 2x-3 \Rightarrow x = \frac{y^2+3}{2}$

$\Rightarrow$ $y = \frac{x^2+3}{2} = f^{-1}(x)$

#.

Thm: (1) f is cont $\Rightarrow f^{-1}$ is cont

(2) f is increasing $\Rightarrow f^{-1}$ is increasing

(3) f is decreasing $\Rightarrow f^{-1}$ is decreasing.

(4) $\boxed{f \text{ is diff at } c \ \& \ f'(c) \neq 0 \Rightarrow f^{-1} \text{ is diff at } f(c)}$

Thm: If f is diff on I and f has an inverse function g

$\Rightarrow g$ is diff at x for which $f'(g(x)) \neq 0$.

Moreover,

$\cancel{g'(x)} =$

$$(f^{-1})'(x) = \left(\cancel{f^{-1}(x)}\right)' = g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0$$

Pf: $\because \ x = f(g(x))$

$\Rightarrow \cancel{x} \ 1 = f'(g(x)) \cdot g'(x)$

$\Rightarrow \cancel{g} \ g'(x) = \dfrac{1}{f'(g(x))}$

#.

Ex: Let $f(x) = \frac{1}{4}x^3 + x - 1$

(a) What is the value of $f^{-1}(x)$ when $x = 3$

(b) . . . . $(f^{-1})'(x)$ when $x = 3$

Sol: $y = f(x) = \frac{1}{4}x^3 + x - 1$

$$\frac{dy}{dx} = \frac{3}{4}x^2 + 1 > 0 \Rightarrow f \nearrow$$

$\therefore$ f has inverse.

$y = \frac{1}{4}x^3 + x - 1$ (不能表示反函數).

$$x \underset{f^{-1}}{\overset{f}{\rightleftarrows}} f(x)$$

Solve $f(x) = 3 \Rightarrow \frac{1}{4}x^3 + x - 1 = 3$

$$\Rightarrow \frac{1}{4}x^3 + x - 4 = 0 \Rightarrow x^3 + 4x - 16 = 0$$

$$\Rightarrow (x-2)(x^2 + 2x + 8) = 0 \Rightarrow x = 2.$$

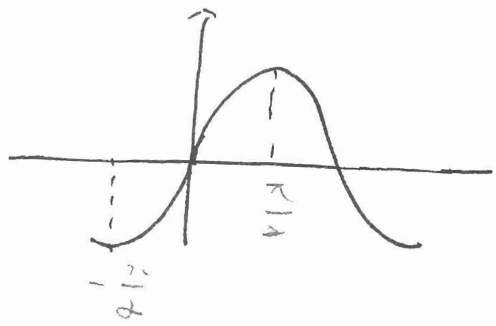
$\therefore$ when $x = 2 \Rightarrow f^{-1}(x) = 3$.

(b)

$$(f^{-1})'(3) = \frac{1}{f'(2)} = \frac{1}{4} \quad \#$$

§ 5.8 Inverse Trigonometric functions and Differentiation.

$Y = \sin x$



on $[-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow Y = \sin x$ has an <u>inverse</u>.

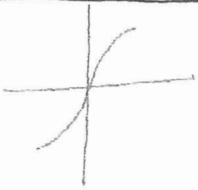
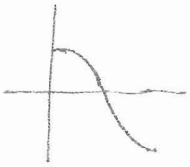
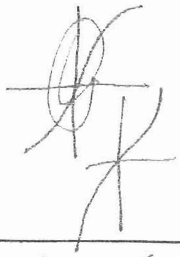
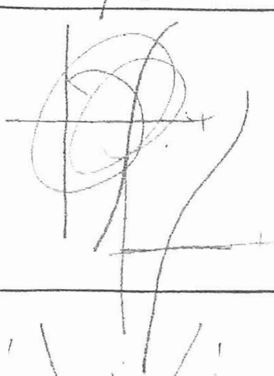
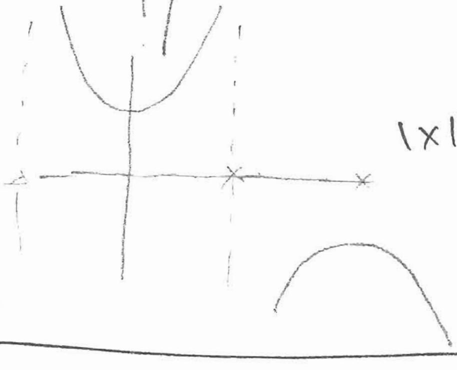
①

		domain	range
$Y = \sin x = f(x)$	$\rightarrow$	$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$	$-1 \le y \le 1$
$Y = \arcsin x = f^{-1}(x)$	$\rightarrow$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

②

$$\boxed{\begin{array}{l} Y = \arcsin x \quad \text{iff} \quad \sin y = x \\ \text{where } -1 \le x \le 1 \text{ and } -\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2} \end{array}}$$

Definition of Inverse Trigonometric functions

function		Domain	Range
$y = \arcsin x \iff x = \sin y$		$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x \iff x = \cos y$		$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x \iff x = \tan y$		$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \operatorname{arccot} x \iff x = \cot y$		$-\infty < x < \infty$	$0 < y < \pi$
$y = \operatorname{arcsec} x \iff x = \sec y$		$\lvert x \rvert \geq 1$	$0 \le y \le \pi,\ y \ne \frac{\pi}{2}$
$y = \operatorname{arccsc} x \iff x = \csc y$		$\lvert x \rvert \geq 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$.

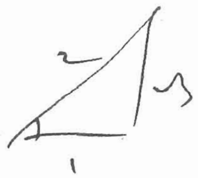
Ex: Evaluate (a) $\arcsin\left(-\frac{1}{2}\right)$

(b) $\arccos 0$

(c) $\arctan\sqrt{3}$

Sol: (a) $y = \arcsin\left(-\frac{1}{2}\right)$

$\Rightarrow \sin y = -\frac{1}{2} \Rightarrow y = -\frac{\pi}{6}$.



(b) $y = \arccos 0$

$\Rightarrow \cancel{\sin} \cos y = 0 \Rightarrow y = \frac{\pi}{2}$

(c) $y = \arctan\sqrt{3} \Rightarrow \tan y = \sqrt{3} \Rightarrow y = \frac{\pi}{3}$

#.

Ex: Solve $\arctan(2x-3) = \frac{\pi}{4}$

Sol: $\tan\left(\arctan(2x-3)\right) = \tan\frac{\pi}{4} = 1$

$\Rightarrow \arctan(2x-3) = \cancel{\tan} \arctan 1$

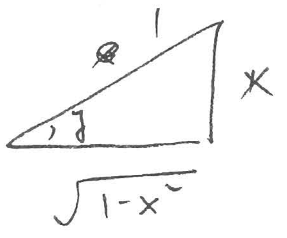
$\Rightarrow 2x-3 = 1 \Rightarrow x = 2$.

Ex: (a) Given $y = \arcsin x$, where $0 < y < \frac{\pi}{2}$,

find $\cos y$.

(b) Given $y = \operatorname{arcsec}\left(\frac{\sqrt{5}}{2}\right)$, find $\tan y$.

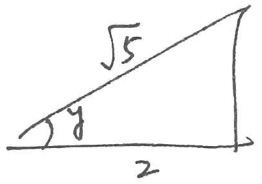
Sol: (a) $\because$ $y = \arcsin x$



$\Rightarrow$ $x = \sin y$

$\Rightarrow$ $\cos y = \sqrt{1-x^2}$.

(b) $\because$ $y = \operatorname{arcsec}\left(\frac{\sqrt{5}}{2}\right)$



$\Rightarrow$ $\frac{\sqrt{5}}{2} = \sec y$

$\therefore$ $\tan y = \frac{1}{2}$

#.

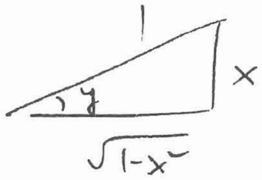
Derivatives of Inverse Trigonometric functions.

Theorem:

$$\begin{cases} \dfrac{d}{dx} \arcsin x = \dfrac{1}{\sqrt{1-x^2}} \\[2ex] \dfrac{d}{dx} \arccos x = \dfrac{-1}{\sqrt{1-x^2}} \\[2ex] \dfrac{d}{dx} \arctan x = \dfrac{1}{1+x^2} \\[2ex] \dfrac{d}{dx} \operatorname{arccot} x = \dfrac{-1}{1+x^2} \\[2ex] \dfrac{d}{dx} \operatorname{arcsec} x = \dfrac{1}{|x|\sqrt{x^2-1}} \\[2ex] \dfrac{d}{dx} \operatorname{arccsc} x = \dfrac{-1}{|x|\sqrt{x^2-1}} \end{cases}$$

proof:

$$y = \arcsin x$$



$$\Rightarrow \quad x = \sin y$$

$$\Rightarrow \quad 1 = \cos y \cdot y' \Rightarrow y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

$$y = \arctan x$$

$$\Rightarrow \quad x = \tan y$$

$$\Rightarrow \quad 1 = \sec^2 y \cdot y'$$

$$\Rightarrow \quad y' = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$y = \operatorname{arcsec} x$

$\Rightarrow x = \sec y$

$\Rightarrow 1 = \sec y \cdot \tan y \cdot y'$

$\Rightarrow y' = \dfrac{1}{\sec y \cdot \tan y} = \dfrac{1}{|x| \cdot \sqrt{x^2-1}}$.

$x^2 - 1 = \sec^2 y - 1$

$= \sqrt{\tan}$

$= \tan^2 y$

$\tan y = \pm\sqrt{x^2-1}$

Ex: (a) $\dfrac{d}{dx} \arcsin\sqrt{x}$

$$= \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}\sqrt{1-x}}.$$

(b) $\dfrac{d}{dx} \operatorname{arcsec} e^{2x}$

$$= \frac{1}{\cancel{e^{2x}} \cdot \sqrt{e^{4x}-1}} \cdot 2 \cdot \cancel{e^{2x}} = \frac{2}{\sqrt{e^{4x}-1}}.$$

Ex: Differentiate $y = \arcsin x + x\sqrt{1-x^2}$

$$\Rightarrow y' = \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \sqrt{1-x^2} + x \cdot \frac{1}{2} \frac{1}{\sqrt{1-x^2}} \cdot (-2)x$$

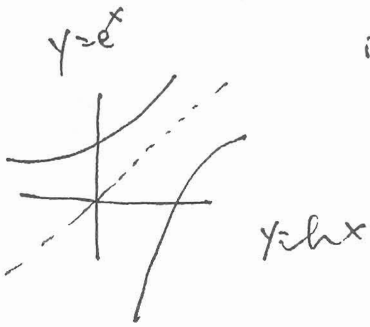
$$= \frac{1}{\sqrt{1-x^2}} + \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}$$

$$= \frac{1+1-x^2-x^2}{\sqrt{1-x^2}} = \frac{2(1-x^2)}{\sqrt{1-x^2}} = 2\sqrt{1-x^2}$$

⊙ $\dfrac{de^x}{dx} = e^x$ ⊙ $\dfrac{d\ln x}{dx} = \dfrac{1}{x}$ $\underline{x>0}$.

$\underline{\int e^x\,dx = e^x + c}$ $\underline{\int \frac{1}{x}\,dx = \ln|x| + C}$

$e^x \xleftrightarrow[\text{inverse}]{} \ln x$



Inverse function ⊙: def: $\boxed{f(g(x)) = g(f(x)) = x}$

$\cancel{x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x)) = x}$

$x \underset{g}{\overset{f}{\rightleftarrows}} f(x)$

⊙ ~~f f^{-1}~~ $\boxed{f^{-1} \text{ unique}}$

⊙ ~~f f^{-1}~~

Existence ~~cont cont~~

⊙ f^{-1} exists $\iff$ f is one-to-one

$\Longleftarrow$ f is increasing (decreasing)

$\Longleftarrow \begin{cases} f' > 0 \\ f' < 0 \end{cases}$

①

f	f^{-1}
cont	cont
increasing	increasing
decreasing	decreasing.
$y=f(x)$	$y=f^{-1}(x)$

$\boxed{\text{圖形 對稱 } x=y}$

diff at c diff at $f(c)$ (if $f'(c)\neq 0$)

& $\boxed{{f^{-1}}'(f(c)) = \dfrac{1}{f'(c)}}$

$$\boxed{\begin{aligned} D(f) &= R(f^{-1}) \\ R(f) &= D(f^{-1}) \end{aligned}}$$

11/23

11/27

g is the inverse of f

$\Longleftrightarrow \quad f(g(x)) = x \quad \& \quad g(f(x)) = x.$

Write: $g = f^{-1}$ $\qquad x \underset{f^{-1}}{\overset{f}{\rightleftarrows}} f(x)$

Prop: (1) if g is the inverse of f

$\Rightarrow$ f is the inverse of g

$(f^{-1})^{-1} = f.$

(2) $D(f) = R(f^{-1})$

$R(f) = D(f^{-1})$

(3) Thm 5.6 f, f^{-1} 對稱於 $x = y$.

(4) f is cont $\Rightarrow$ f^{-1} is cont

(5) f is increasing (decreasing) $\Rightarrow$ f^{-1} is increasing (decreasing)

(6) f is diff at c & $f'(c) \neq 0$

$\Rightarrow$ f^{-1} is diff at $f(c)$

$$\boxed{(f^{-1})'(f(c)) = \frac{1}{f'(c)}}$$

Theorem: if f is diff on I and f has an inverse function g

$\Rightarrow$ g is diff at x for which $f'(g(x)) \neq 0$.

Moreover, $$\boxed{(f^{-1})'(x) = g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0}$$

pf:

$$\because \quad x = f(g(x))$$

$$\Rightarrow \quad 1 = f'(g(x)) \cdot g'(x)$$

$$\Rightarrow \quad g'(x) = \frac{1}{f'(g(x))} \quad \#$$

Existence:

Theorem:
(a) A function has an inverse

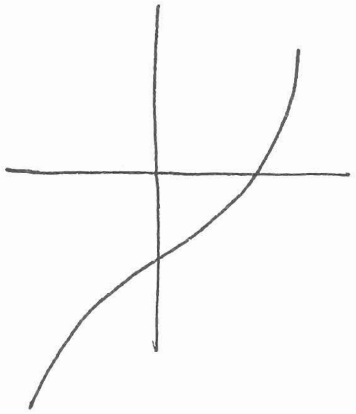
$\Leftrightarrow$ it is one-to-one.

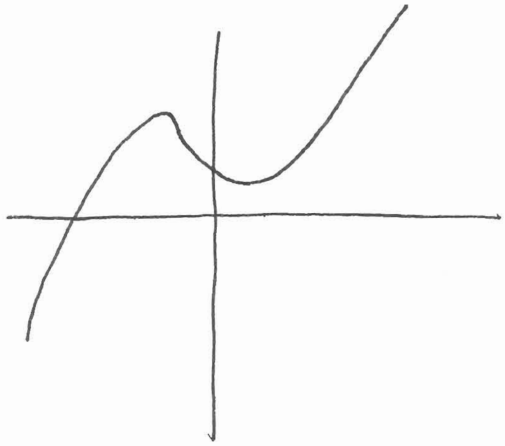
(b) if f ~~has~~ is strictly monotonic (increasing / decreasing),

$\Rightarrow$ it is one-to-one.

Ex: Which of the functions has an inverse

(a) $f(x) = x^3 + x - 1$ (b) $f(x) = x^3 - x + 1$





Ex: Find the inverse of $f(x) = \sqrt{2x-3}$

Sol: Let $y = f(x) = \sqrt{2x-3}$

$\Rightarrow D(f) = \{x \,;\, 2x-3 \geq 0\}$.

Consider $\dfrac{dy}{dx} = \dfrac{1}{2}(2x-3)^{-\frac{1}{2}} \cdot 2 = \dfrac{1}{\sqrt{2x-3}} \geq 0$ on $D(f)$

$\Rightarrow f$ is increasing

$\Rightarrow f^{-1}$ exists.

$y = f(x) = \sqrt{2x-3} \Rightarrow y^2 = 2x-3$

$\Rightarrow x = \dfrac{y^2+3}{2} \Rightarrow y = \dfrac{x^2+3}{2}$

$\Rightarrow f^{-1}(x) = \dfrac{x^2+3}{2}$ #.

Ex: Let $f(x) = \frac{1}{4}x^3 + x - 1$

(a) What is the value of $f^{-1}(x)$ when $x=3$

(b) 〃 〃 〃 〃 〃 $(f^{-1})'(x)$ when $x=3$.

Sol:

$$y = f(x) = \frac{1}{4}x^3 + x - 1 \qquad D(f) = \mathbb{R}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{4}x^2 + 1 > 0 \Rightarrow f \nearrow$$

$$\Rightarrow f^{-1} \text{ exists.}$$

$$x \underset{f^{-1}}{\overset{f}{\rightleftarrows}} f(x)$$

(a)

Solve $f(x) = 3 \Rightarrow \frac{1}{4}x^3 + x - 1 = 3$

$$\Rightarrow x = 2.$$

$\therefore$ $\cancel{\text{when } x=2 \Rightarrow f^{-1}(x)=3}$. $f^{-1}(3) = 2$

(b)

$$(f^{-1})'(3) = \frac{1}{f'(2)} = \frac{1}{\frac{3}{4}\cdot 4 + 1} = \frac{1}{4}.$$

§. 5.4 Exponential functions : Diff. integral

natural logarithmic function $\xleftrightarrow[\text{inverse}]{}$ natural exponential function

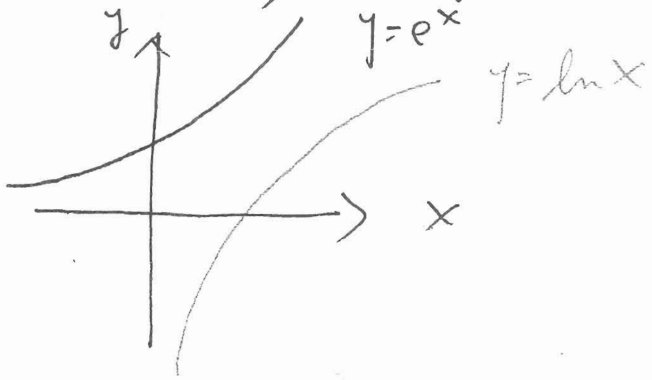
$y = \ln x$ $\qquad\qquad$ $y = e^x$

$$\ln e^x = x, \quad e^{\ln x} = x.$$

$$\therefore \boxed{y = e^x \iff x = \ln y}$$

Prop: Let $f(x) = e^x$

(1) $D(f) = \mathbb{R}$, $R(f) = (0, \infty)$



(2) $f(x) = e^x$ is cont, increasing, one-to-one

(3) concave up

(4) $\lim\limits_{x\to-\infty} e^x = 0$, $\lim\limits_{x\to\infty} e^x = \infty$.

Ex: Solve $7 = e^{x+1}$

Sol: $\ln 7 = \ln e^{x+1}$

$\Rightarrow \ln 7 = x+1 \Rightarrow x = \ln 7 - 1$.

Ex: Solve $\ln(2x-3) = 5$.

Sol $e^{\ln(2x-3)} = e^5$

$\Rightarrow 2x-3 = e^5 \Rightarrow x = \frac{e^5+3}{2}$

#.

Theorem:

(a) $\begin{cases} \frac{d}{dx} e^x = e^x \\ \frac{d}{dx} e^u = e^u \cdot \frac{du}{dx} \end{cases}$ (u is a diff fn of x).

(b) $\int e^x \, dx = e^x + C$.

Ex: (a) $\frac{d}{dx} e^{2x-1} = e^{2x-1} \cdot 2$

(b) $\frac{d}{dx} e^{-\frac{3}{x}} = e^{-\frac{3}{x}} \cdot \left(\frac{3}{x^2}\right)$.

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Ex: $\int e^{3x+1} dx \qquad \int e^{3x+1} d(3x+1)$

$= \int e^{3x+1} d(3x+1) \cdot \frac{1}{3}$

$= \frac{1}{3} e^{3x+1} + C$.

Ex: $\int 5x e^{-x^2} dx$

$= 5\int e^{-x^2} x \, dx$

$= 5\int e^{-x^2} dx^2 \cdot \frac{1}{2}$

$= \frac{5}{2} \int e^{-x^2} d(-x^2)(-1)$

$= -\frac{5}{2} \cdot e^{-x^2} + C$

#

ex:

(a) $\displaystyle\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$

(b) $\displaystyle\int \sin x\, e^{\cos x}\,dx$

(c) $\displaystyle\int_0^1 e^{-x}\,dx$

(d) $\displaystyle\int_0^1 \frac{e^x}{1+e^x}\,dx$

(e) $\displaystyle\int_{-1}^{0} e^x \cos e^x\,dx$

(f) $\displaystyle\int \frac{1}{1+e^x}\,dx$

$u = 1+e^x$

$\dfrac{du}{dx} = e^x$

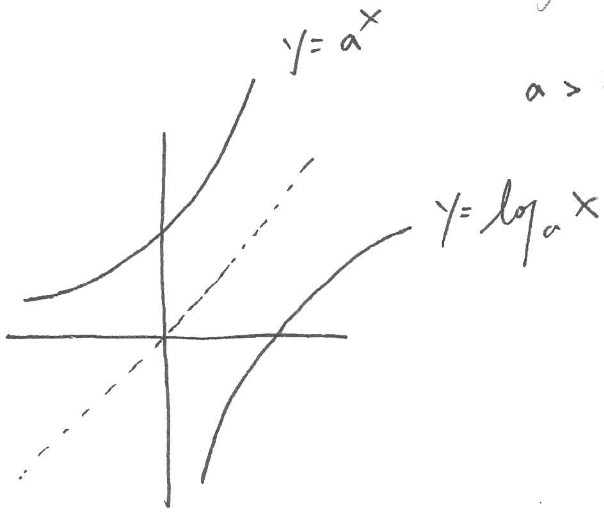
$$= \int \frac{1}{u}\cdot\frac{1}{e^x}\,du$$

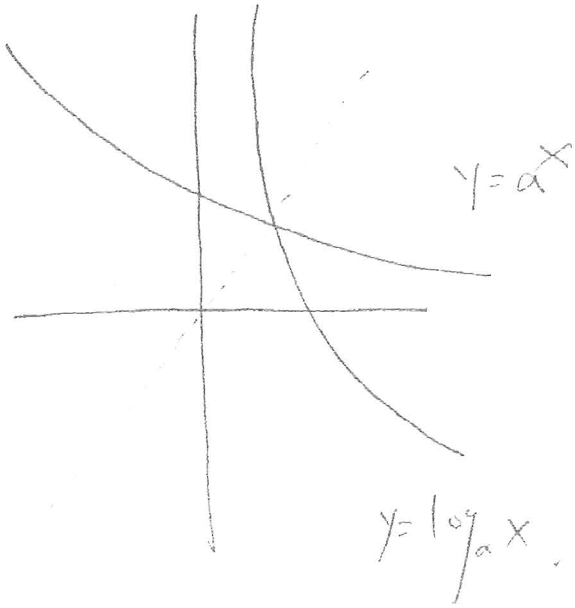
$$= \int \frac{1}{u}\cdot\frac{1}{(u-1)}\,du$$

$$= \int \left(\frac{1}{u-1} - \frac{1}{u}\right) du$$

§ 5.5 Bases Other than e and Applications.

$$\begin{cases} e^x \longrightarrow a^x & \text{exponential fn to the base } a \quad a>0\ (a\neq 1) \\ \updownarrow \text{inverse} \qquad \updownarrow \text{inverse} & \\ \ln x \longrightarrow \log_a x & \text{logarithmic function to the base } a. \end{cases}$$





Prop:

$$\begin{cases} y = a^x \iff x = \log_a y. \\ a^{\log_a x} = x \quad \text{for } x > 0 \\ \log_a a^x = x \quad \forall x. \end{cases}$$

Ex: Solve for x

(a) $3^x = \frac{1}{81}$ (b) $\log_2 x = -4$

Sol: (a) $3^x = \frac{1}{81}$

$\Rightarrow \log_3 3^x = \log_3 \frac{1}{81} = \log_3 3^{-4}$

$\Rightarrow x = -4$

(b) $\log_2 x = -4$

$\Rightarrow 2^{\log_2 x} = 2^{-4}$

$\Rightarrow x = \frac{1}{16}$ #

$e^{\ln x} = x.$

$e^{x \ln a} = a^x$

$e^{x\ln a} \cdot \frac{1}{x \ln a} \cdot = a^x \cdot$

Theorem: $\cancel{\frac{d}{dx} a^x}$

$$\begin{cases} \frac{d}{dx} e^x = e^x \\ \frac{d}{dx} \ln x = \frac{1}{x} \quad (x>0) \end{cases} \qquad \begin{cases} \frac{d}{dx} a^x = \ln a \, a^x \\ \frac{d}{dx} \log_a x = \frac{1}{\ln a} \cdot \frac{1}{x} \end{cases}$$

pf: $\begin{cases} \boxed{a^x = e^{(\ln a) x}} \\ \log_a x = \frac{\ln x}{\ln a} \end{cases}$

Ex: (a) $\frac{d}{dx} 2^x$

$= \ln 2 \cdot 2^x$

(b) $\frac{d}{dx} 2^{3x} = \ln 2 \cdot 2^{3x} \cdot 3 = 3 \ln 2 \; 2^{3x}$

(c) $\frac{d}{dx} \log_{10} \cos x = \frac{1}{\ln 10} \cdot \frac{1}{\cos x} \cdot (-\sin x)$

$= \frac{-1}{\ln 10} \tan x$.

$$\begin{cases} \int a^x \, dx = \frac{1}{\ln a} a^x + C . \\ \int \log_a x \, dx = ?? \end{cases}$$

Ex: Evaluate $\int 2^x \, dx$

$= \frac{1}{\ln 2} \cdot 2^x + C$.

Ex: (a) $\frac{d}{dx} e^e = 0$

(b) $\frac{d}{dx} e^x = e^x$

(c) $\frac{d}{dx} x^e = e x^{e-1}$

(d) $\frac{d}{dx} x^x = ?\,?$

sol. $y = x^x$

$\Rightarrow \ln y = \ln x^x$

$\Rightarrow \ln y = x \ln x$

$\Rightarrow \frac{1}{y} y' = \ln x + 1 \Rightarrow y' = y(\ln x + 1)$

$= x^x (\ln x + 1)$ #.

Thm: $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} \left(\frac{x+1}{x}\right)^x = e$.

Derivatives of Inverse Trigonometric functions.

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Theorem:

$$\begin{cases} \dfrac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}} \\[2ex] \dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2} \\[2ex] \dfrac{d}{dx}\sec^{-1}x = \dfrac{1}{|x|\sqrt{x^2-1}} \end{cases}$$

Ex: (a) $\dfrac{d}{dx}\sin^{-1}\sqrt{x}$

$$= \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2}\frac{1}{\sqrt{x}}$$

(b) $\dfrac{d}{dx}\,\mathrm{arc\,sec}\, e^{2x} = \dfrac{d}{dx}\sec^{-1} e^{2x}$

$$= \frac{1}{e^{2x}\sqrt{e^{4x}-1}} \cdot 2\cancel{e^{2x}}$$

$$= \frac{2}{\sqrt{e^{4x}-1}}$$

Ex: Differentiate $y = \sin^{-1}x + x\sqrt{1-x^2}$

Sol: $y' = \dfrac{1}{\sqrt{1-x^2}} + \sqrt{1-x^2} + x \cdot \dfrac{1}{2}\dfrac{1}{\sqrt{1-x^2}}(-2x)$

$$= \frac{1}{\sqrt{1-x^2}} + \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}$$

$$= \frac{1+1-x^2-x^2}{\sqrt{1-x^2}} = \frac{2(1-x^2)}{\sqrt{1-x^2}} = 2\sqrt{1-x^2}$$

\#

Integration:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c$$

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}|x| + C.$$

$$\Rightarrow \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\,d\cancel{x}\left(\frac{x}{a}\right)$$

$$= \sin^{-1}\left(\frac{x}{a}\right) + c.$$

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{1+\left(\frac{x}{a}\right)^2}\,d\left(\frac{x}{a}\right)\cdot\frac{1}{a}$$

$$= \frac{1}{a}\tan^{-1} x + c$$

$$\int \frac{1}{\cancel{a}\,x\sqrt{x^2-a^2}}\,dx = \int \frac{1}{x\cdot a\sqrt{\left(\frac{x}{a}\right)^2-1}}\,dx$$

$$= \int \frac{\cancel{a}}{a\cdot\frac{x}{a}\sqrt{\left(\frac{x}{a}\right)^2-1}}\cdot d\left(\frac{x}{a}\right)$$

$$= \frac{1}{a}\int \frac{1}{\frac{x}{a}\sqrt{\left(\frac{x}{a}\right)^2-1}}\,d\left(\frac{x}{a}\right)$$

$$= \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + c$$

#.

Ex: (a) $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\left(\frac{x}{2}\right) + C.$

(b) $\displaystyle\int \frac{dx}{2+9x^2} = \int \frac{1}{2\left(1+\frac{9}{2}x^2\right)}\,dx$

$$= \int \frac{1}{1+\left(\frac{3}{\sqrt{2}}x\right)^2}\, d\left(\frac{3}{\sqrt{2}}x\right)\cdot\frac{\sqrt{2}}{3}\cdot\frac{1}{2}$$

$$= \frac{1}{3\sqrt{2}}\cdot\tan^{-1}\frac{3}{\sqrt{2}}x + C.$$

(c) $\displaystyle\int \frac{dx}{x\sqrt{4x^2-9}} =$

Ex: Evaluate $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$

$$\underset{\frac{du}{dx}=e^x}{\overset{u=e^x}{=}} \int \frac{1}{\sqrt{u^2-1}}\cdot\frac{1}{u}\,du$$

$$= \int \frac{1}{u}\frac{1}{\sqrt{u^2-1}}\,du = \sec^{-1}|u| + C$$

$$= \sec^{-1}e^x + C$$

$t = \sqrt{u}$

$\frac{dt}{du} = \frac{1}{2}\frac{1}{\sqrt{u}}$

$\displaystyle\int \frac{1}{t^2+1}\,dt$

$= \tan^{-1}t + C$

$= \tan^{-1}\sqrt{e^{2x}-1} + C$

Ex: Evaluate $\displaystyle\int \frac{x+2}{\sqrt{4-x^2}}\,dx$

$$= \int \frac{x}{\sqrt{4-x^2}}\,dx + \int \frac{2}{\sqrt{4-x^2}}\,dx$$

$$\underset{\frac{du}{dx}=-2x}{\overset{u=4-x^2}{=\!=}} \int \frac{1}{\sqrt{u}} \cdot \left(-\frac{1}{2}\right) du + \int \frac{\cancel{2}}{\cancel{2}\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx$$

$$= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du + \int \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,d\left(\frac{x}{2}\right)\cdot 2$$

$$= -\frac{1}{2}\cdot 2u^{\frac{1}{2}} + 2\sin^{-1}\left(\frac{x}{2}\right) + C$$

$$= -(4-x^2)^{\frac{1}{2}} + 2\sin^{-1}\left(\frac{x}{2}\right) + C$$

#

Ex: $\displaystyle\int \frac{1}{x^2-4x+7}\,dx$

$$= \int \frac{1}{x^2-4x+4+3}\,dx$$

$$= \int \frac{1}{(x-2)^2+3}\,dx$$

$$= \int \frac{1}{3\left[\left(\frac{x-2}{\sqrt{3}}\right)^2+1^2\right]}\,dx$$

$$= \int \frac{1}{3\left[\left(\frac{x-2}{\sqrt{3}}\right)^2+1^2\right]}\, d\left(\frac{x-2}{\sqrt{3}}\right)\cdot\sqrt{3}$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x-2}{\sqrt{3}}\right)+C$$ #.

Ex: (a) $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}|x|+C$

(b) $\int \frac{x}{\sqrt{x^2-1}}\,dx$

$$\underset{\frac{du}{dx}=2x}{\overset{u=x^2-1}{=\!=\!=}} \int \frac{1}{\sqrt{u}}\,\frac{1}{2}\,du = u^{+\frac{1}{2}}+C$$

$$= \sqrt{x^2-1}+C$$

(c) $\int \frac{1}{\sqrt{x^2-1}}\,dx$

Ex: (a) $\int \frac{dx}{x\ln x} = \int \frac{1}{\ln x}\,d\ln x = \ln|\ln x|+C$ #.

(b) $\int \frac{\ln x}{x}\,dx = \int \ln x\, d(\ln x) = \frac{(\ln x)^2}{2}+C$.

(c) $\int \ln x\,dx = ?$