Ch 4. Integration

11/6.

§ 4.1 : Antiderivatives & Indefinite Integration

$F(x) = \cancel{3}x^3 \longrightarrow$ derivative : $F'(x) = 3x^2$

antiderivatives : $x^3 \longleftarrow f(x) = 3x^2$

$x^3 + C$.

Theorem: if F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I iff G is of the form

$$G(x) = F(x) + C, \quad \forall x \in I$$

where C is a constant.

$F(x) = x^3 \longrightarrow \underline{\underline{\dfrac{dF}{dx} = 3x^2}}$

$$\boxed{\int 3x^2\,dx = x^3 + C}$$

indefinite integration

(anti differentiation)

◎

$$\boxed{\int \underline{f}(x)\,dx = F(x) + C} \qquad F'(x) = f(x)$$

integrand variable of integration

◎

$$\int F'(x)\,dx = F(x) + C.$$

◎

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x).$$

Diff formula:

$$\frac{d}{dx} C = 0$$

$$\frac{d}{dx} k f(x) = k f'(x)$$

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x.$$

⋮

$$\left\{ \begin{aligned} &(f \pm g)' = f' \pm g' \\ &(fg)' = f'g + fg' \\ &\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \\ &(f \circ g)' = f'(g(x)) \cdot g'(x) \end{aligned} \right.$$

Integration formula:

$$\cancel{\int 0 \, dx = C} \quad \int k \, dx = kx + C$$

$$\int k f(x) \, dx = k \int f(x) \, dx$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \tan x \, dx =$$

$$\int \cot x \, dx$$

$$\int \sec x \, dx$$

$$\int \csc x \, dx$$

$$\int (f \pm g) \, dx = \int f \, dx \pm \int g \, dx$$

$$\cancel{\int fg \, dx}$$

$$\cancel{\int f/g}$$

Ex: (a) $\int 3x\,dx = 3\int x\,dx = 3\left(\frac{x^2}{2}\right) + C$

(b) $\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = -\frac{1}{2}x^{-2} + C$

(c) $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}} + C$

(d) $\int 2\sin x\,dx = 2\int \sin x\,dx = 2(-\cos x) + C$

$= -2\cos x + C$

Ex: (a) $\int dx = x + C$.

(b) $\int (x+2)\,dx = \int x\,dx + \int 2\,dx$

$= \frac{x^2}{2} + 2x + C$.

(c) $\int (3x^4 - 5x^2 + x)\,dx$

$= \int 3x^4dx - \int 5x^2\,dx + \int x\,dx$

$= 3\int x^4\,dx - 5\int x^2\,dx + \int x\,dx$

$= \frac{3}{5}x^5 - \frac{5}{3}x^3 + \frac{x^2}{2} + C.$

Ex: $\int \frac{x+1}{\sqrt{x}} \, dx$

$= \int \frac{x}{\sqrt{x}} \, dx + \int \frac{1}{\sqrt{x}} \, dx$

$= \int x^{\frac{1}{2}} \, dx + \int x^{-\frac{1}{2}} \, dx$

$= \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$.

Ex: $\int \frac{\sin x}{\cos^2 x} \, dx$

$= \int \frac{1}{\cos^2 x} (-d\cos x)$

$= \frac{1}{\cos x} + C = \sec x + C$.

$= \int \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \, dx$

$= \int \sec x \tan x \, dx$

$= \sec x + C$

#.

Ex: Find the general solution of

$$F'(x) = \frac{1}{x^2}, \quad x > 0$$

and find the particular solution that satisfies the initial condition $F(1) = 0$.

Sol: (1)

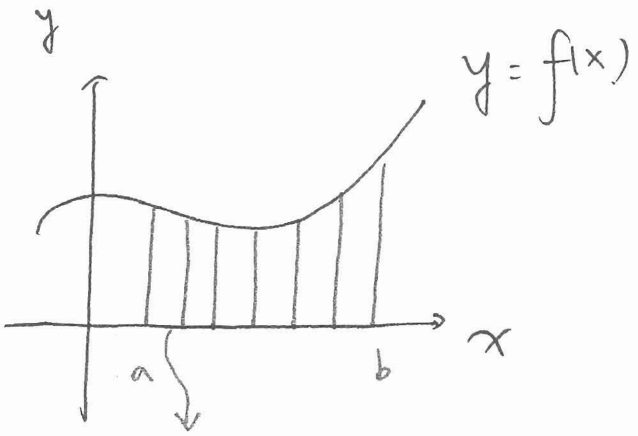
$$F(x) = \int F'(x)\,dx = \int \frac{1}{x^2}\,dx = \int x^{-2}\,dx$$

$$= -x^{-1} + C = -\frac{1}{x} + C$$

(2) $F(1) = -1 + C = 0 \Rightarrow C = 1$

$$\therefore F(x) = -\frac{1}{x} + 1, \quad x > 0.$$

§ 4.2 Area :



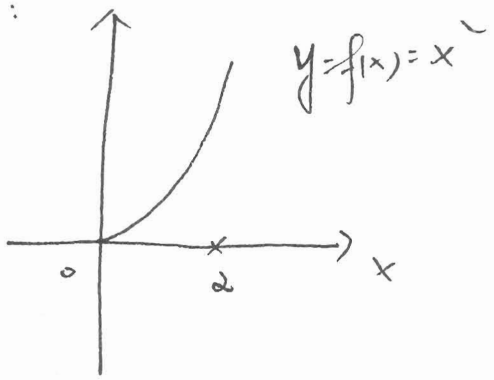
$$\Delta X = \frac{b-a}{n}$$

$$\begin{cases} a = x_0 < x_1 < x_2 < \cdots < x_n = b. \\ f(m_i) = \text{minimum value of } f(x) \text{ in } i\text{-th subinterval} \\ f(M_i) = \text{maximum value of } f(x) \text{ in } i\text{-th subinterval} \end{cases}$$

$$\begin{cases} s(n) = \sum_{i=1}^{n} f(m_i)\Delta x & : \text{Lower sum} \\ S(n) = \sum_{i=1}^{n} f(M_i)\Delta x & : \text{Upper sum}. \end{cases}$$

Ex: Find the upper and lower sums for the region bounded by the graph of $f(x) = x^2$ and the x-axis between $x=0$ and $x=2$.

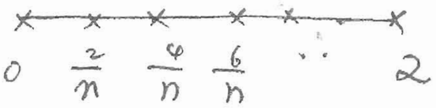
sol:



$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

$\because$ $f'(x) = 2x > 0$ on $[0, \infty)$

$\Rightarrow f$ is increasing



$\therefore$
$$\begin{cases} m_i = 0 + (i-1)\left(\frac{2}{n}\right) = \frac{2(i-1)}{n} \\ M_i = 0 + i\left(\frac{2}{n}\right) = \frac{2}{n} i \end{cases}$$

Hence

- Low sum: $s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x$

$$= \sum_{i=1}^{n} \left(\frac{2(i-1)}{n}\right)^2 \cdot \frac{2}{n} = \sum_{i=1}^{n} \frac{8}{n^3}(i-1)^2$$

$$= \sum_{i=1}^{n} \frac{8}{n^3}(i^2 - 2i + 1)$$

$$= \frac{8}{n^3}\left[\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \right]$$

$$= \frac{8}{n^3}\left\{ \frac{n(n+1)(2n+1)}{6} - 2\left[\frac{n(n+1)}{2}\right] + n \right\}$$

$$= \frac{4}{3n^3}(2n^3 - 3n^2 + n)$$

$$= \frac{8}{3} - \frac{4}{n} + \frac{4}{3n^2}.$$

upper sum :

$$S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta X$$

$$= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\cdot\frac{2}{n}$$

$$= \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^2\cdot\frac{2}{n}$$

$$= \sum_{i=1}^{n} \frac{8}{n^3} i^2$$

$$= \cancel{\sum_{i=1}^{n}} \frac{8}{n^3} \; \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{4}{3n^3}(2n^3 + 3n^2 + n)$$

$$= \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}.$$

$$\begin{cases} s(n) < S(n) \\ \lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n). \end{cases}$$

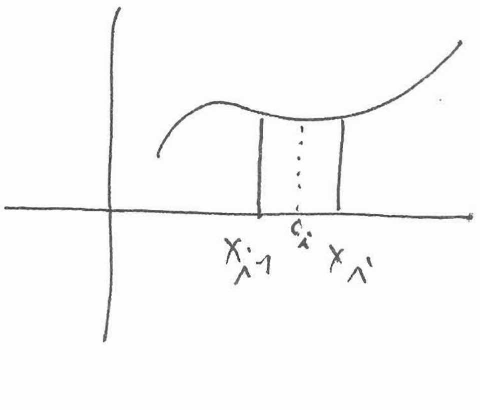
Thm 4.3: Let f be cont and non-negative on $[a, b]$. The limit as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\lim_{n \to \infty} s(n) = \lim_{n \to \infty} S(n).$$

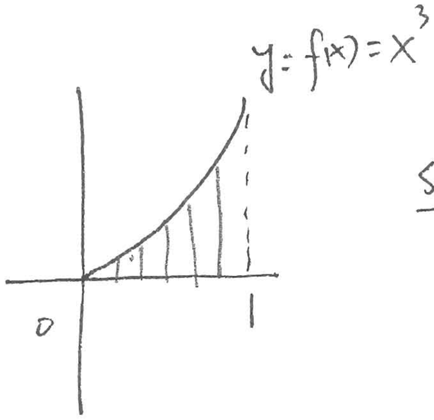
Def: Let f be cont and non-negative on $[a, b]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

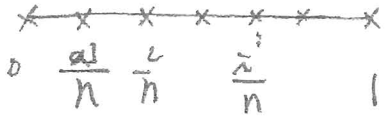
$$\text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\,\Delta x, \quad x_{i-1} \le c_i \le x_i$$

where $\Delta x = \dfrac{b-a}{n}$



Ex: Find the area of the region bounded by the graph $f(x) = x^3$, the x-axis, and the vertical lines $x=0$ and $x=1$.





Sol:

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty} \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]$$

$$= \lim_{n\to\infty} \frac{n^4 + 2n^3 + n^2}{4n^4} = \frac{1}{4}$$

#.

$x^3 \xrightarrow{\text{diff}}$ derivative $\underline{\underline{3x^2}}$

$\frac{d}{dx}\int f(x)\,dx = f(x)$

antiderivative $x^3 + C$ $\xleftarrow{\text{integral}}$

$\frac{d}{dx}x^3 = 3x^2$

antiderivative, indefinite integral,

$$\boxed{\int 3x^2\,dx = x^3 + C}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\boxed{\begin{aligned} &\int (f \pm g)\,dx = \int f \pm \int g \\ &\int (af)\,dx = a\int f\,dx \end{aligned}}$$

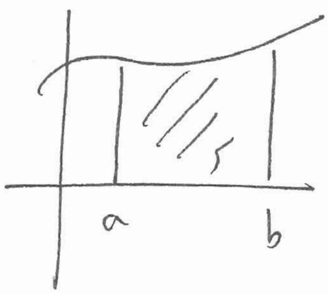
~~$\int$ trian~~

$\int \sin x\,dx$

$\int \cos x\,dx$

$\vdots$

$\int \csc x\,dx$



$y = f(x)$ $f(x) \geq 0$, cont on $[a, b]$.

upper-sum

Lower-sum

$x_0 = a, x_1, x_2, \ldots, b$

$\boxed{x_i = x_0 + i\,\frac{b-a}{n}}$

upper-sum : $\lim_{n\to\infty}\sum_{i=1}^{n} f(M_i)\,\Delta x = S(n)$

lower-sum : $\lim_{n\to\infty}\sum_{i=1}^{n} f(m_i)\,\Delta x = s(n)$

$\int_{-1}^{1} x^2\,dx$ $\quad F(x) = \int_{x}^{x+2} (4t+1)\,dt$

$$\boxed{\text{if } f \geq 0 \text{ and cont. on } [a,b] \Rightarrow \underline{\underline{S(n)}} = \lim \underline{\underline{s(n)}}}$$

$$\underline{\underline{\text{Area}}} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x \quad , \quad \text{where } \underline{\underline{x_{i-1} \leq c_i \leq x_i}}$$

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§ 4.3: Riemann sums and Definite integrals.

Def: Let f be defined on $[a,b]$, and let Δ be a partition of $[a,b]$ given by

$a = x_0 < x_1 < \cdots < x_n = b$. where

Δx_i is the length of the i-th subinterval of.

if c_i is any point in the i-th subinterval

then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i \quad , \quad x_i \leq c_i \leq c_i \quad \text{is called}$$

a Riemann sum of f for the partition Δ

Ref: if f is defined on $[a.b]$ and the $\lim\limits_{\|\Delta\| \to 0} \sum\limits_{i=1}^{n} f(c_i) \Delta x_i$ exists, then f is integrable on $[a.b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i = \int_a^b f(x)\, dx .$$

Thm: if f is cont on $[a.b]$

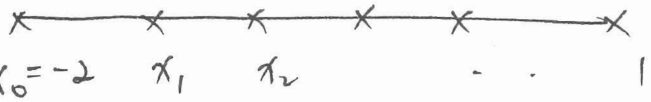
$\Rightarrow$ f is integrable on $[a.b]$.

$\Leftarrow$??

Ex: Evaluate the definite integral $\int_{-2}^{1} 2x\,dx$

Sol: Let $\begin{cases} f(x) = 2x \\ \Delta x_i = \Delta x = \dfrac{1-(-2)}{n} = \dfrac{3}{n}. \end{cases}$

$\therefore\ x_i = -2 + \frac{3}{n} i$



Take $c_i = -2 + \frac{3}{n} i$

$$\therefore \int_{2}^{1} 2x\,dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

$$= \cancel{\lim_{\|\Delta\|\to 0}} \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-2+\frac{3}{n}i\right) \frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} 2\left(-2+\frac{3}{n}i\right) \frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} \left(-2+\frac{3}{n}i\right)$$

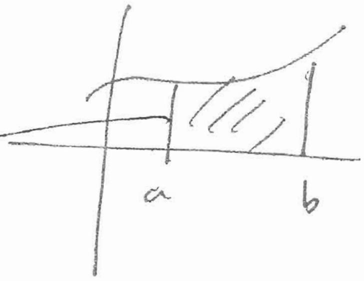
$$= \lim_{n\to\infty} \frac{\cancel{6}^{3}}{n} \left[(-2n) + \frac{3}{\cancel{n}} \cdot \frac{n(n+1)}{\cancel{2}}\right]$$

$$= \lim_{n\to\infty} \left(-12 + 9 + \frac{9}{n}\right)$$

$$= -3$$

#

Thm: if f is cont and $f \geq 0$ on $[a, b]$. Then



$$\boxed{\text{Area} = \int_a^b f(x)\,dx}$$

⓪ $\int_a^a f(x)\,dx = 0$.

② $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

③ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

$c \in [a, b]$

④ $\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$

⑤ $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

⑥

② if f is integrable and $f \geq 0$ on $[a,b]$

$$\Rightarrow \quad 0 \leq \int_a^b f(x)\,dx$$

③ if f, g are integrable on $[a,b]$

and $f(x) \leq g(x) \quad \forall\, x \in [a,b]$

$$\Rightarrow \quad \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

9. (10%) Find the greatest and smallest values that the function

$$f(x,y) = xy$$

takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

Review :

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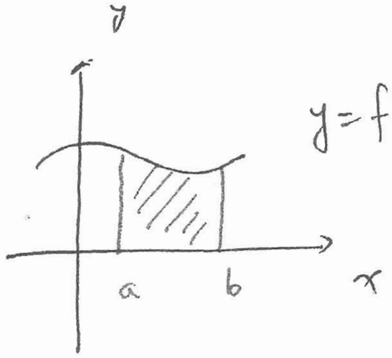
① $\int f(x)\,dx$ → To find the anti-derivative

→ Indefinite integral

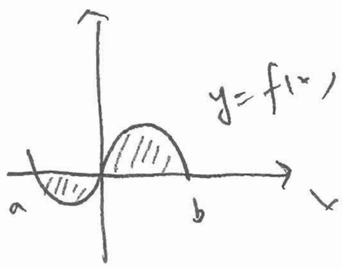
② 

$f \geq 0$, cont on $[a,b]$

$\lim_{n\to\infty} S(n) = \lim_{n\to\infty} s(n) = \text{Area}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$

$x_{i-1} \leq c_i \leq x_i$

③ 

f ~~on [a~~ cont on $[a,b]$

$\Rightarrow$ Riemann sum 的極限存在.

$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$ exists

$\Delta = \{a = x_0, x_1, \cdots, x_n\}$

is any partition on $[a,b]$

$\Longleftrightarrow$ f is integrable on $[a,b]$.

$$\boxed{\int_a^b f(x)\,dx = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i}$$

① $\int_a^a f(x)\,dx = 0$

② $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

③ $\int_a^b f(x)\,dx \geq 0$ if $f \geq 0$ on $[a,b]$

④ $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

if $f(x) \leq g(x)$ on $[a,b]$

⑤ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

§ 4.4: The fundamental Theorem of Calculus.

Theorem: (The fundamental Theorem of Calculus)

if f is cont. on $[a, b]$, and F is an anti-derivative of f on $[a, b]$, then

$$\underline{\int_a^b f(x)\,dx = F(b) - F(a)}.$$

Ex: $\displaystyle\int_1^2 (x^2-3)\,dx = \left(\frac{x^3}{3} - 3x\right)\Bigg|_1^2$ $\qquad F(x) = \frac{x^3}{3} - 3x$

$$= \underbrace{\left(\frac{8}{3} - 6\right)}_{F(2)} - \underbrace{\left(\frac{1}{3} - 3\right)}_{F(1)}$$

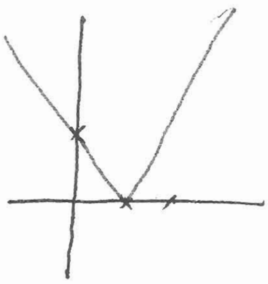
$$= -\frac{2}{3}.$$

Ex: $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x\,dx = \tan x \Big|_0^{\frac{\pi}{4}}$

$$= 1.$$

Ex: Evaluate $\displaystyle\int_0^2 |2x-1|\,dx$

$$|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \geq 0 \\ -(2x-1) & \text{if } 2x-1 \leq 0. \end{cases}$$



$$\Rightarrow \quad |2x-1| = \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2} \\ -2x+1 & \text{if } x \leq \frac{1}{2} \end{cases}$$

$$\therefore \int_0^2 |2x-1|\,dx$$

$$= \int_0^{\frac{1}{2}} |2x-1|\,dx + \int_{\frac{1}{2}}^2 |2x-1|\,dx$$

$$= \int_0^{\frac{1}{2}} (-2x+1)\,dx + \int_{\frac{1}{2}}^2 (2x-1)\,dx$$

$$= (-x^2+x)\Big|_0^{\frac{1}{2}} + (x^2-x)\Big|_{\frac{1}{2}}^2$$

$$= \frac{5}{2}.$$

Ex: $\boxed{\text{p. 276 ex 3}}$

① Theorem : $\boxed{\text{Mean Value Theorem for integrals}}$

if f is cont on $[a, b]$

$\Rightarrow \exists \; c \in [a, b]$, s.th.

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

$\Downarrow$

$\boxed{\text{the average value of } f}$

~~Df~~

$\cancel{f(c) =}$

②

the average value of f ~~is~~ on $[a, b]$ ~~b-a~~

is $\boxed{\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx}$

The Second fundamental Thm of Calculus.

$\boxed{\int_a^b f(x)\,dx}$ $\boxed{F(x) = \int_a^x f(t)\,dt}$

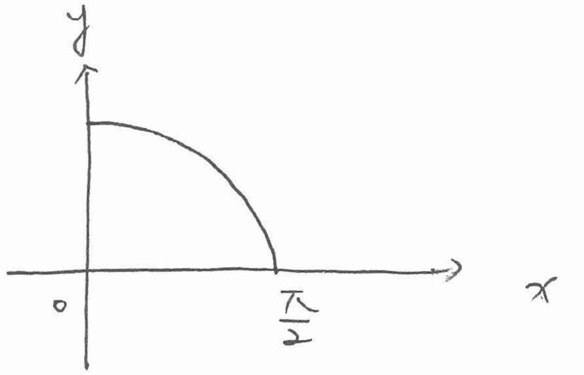
$F'(x) = ??$

F ??

Ex: Evaluate the function

$$F(x) = \int_0^x \cos t \, dt .$$

at $x=0$, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$



$$F(x) = \int_0^x \cos t \, dt$$
$$= \sin t \Big|_0^x$$
$$= \sin x - \sin 0$$
$$= \sin x$$

$$F(0) = \int_0^0 \cos t \, dt = 0$$

$$F\left(\frac{\pi}{6}\right) = \int_0^{\frac{\pi}{6}} \cos t \, dt = \frac{1}{2}$$

$$F\left(\frac{\pi}{4}\right) = \int_0^{\frac{\pi}{4}} \cos t \, dt = \frac{\sqrt{2}}{2}$$

$$F\left(\frac{\pi}{3}\right) = \int_0^{\frac{\pi}{3}} \cos t \, dt = \frac{\sqrt{3}}{2}$$

$$F\left(\frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}} \cos t \, dt . = 1.$$

Thm: $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ if f is cont on $[a,b]$

Ex: Evaluate $\dfrac{d}{dx}\left[\displaystyle\int_0^x \sqrt{t^2+1}\,dt\right]$

$$= \sqrt{x^2+1}$$

Ex: ~~Evaluate~~ Find the derivative of

$$F(x) = \int_{\frac{\pi}{2}}^{x^3} \cos t\,dt.$$

Sol: ① $F(x) = \displaystyle\int_{\frac{\pi}{2}}^{x^3} \cos t\,dt = \sin t \Big|_{\frac{\pi}{2}}^{x^3}$

$$= \sin x^3 - 1.$$

② ~~Let $u = x^3$.~~ $\Rightarrow F'(x) = 3x^2 \cos x^3$

② Let $u = x^3$, $g(u) = \displaystyle\int_{\frac{\pi}{2}}^{u} \cos t\,dt$

$$x \xrightarrow{u} x^3 \xrightarrow{g} \int_{\frac{\pi}{2}}^{x^3} \cos t\,dt.$$

$\therefore$ $F(x) = \displaystyle\int_{\frac{\pi}{2}}^{x^3} \cos t\,dt = g(u(x))$

By chain rule

$$F'(x) = \frac{d\,g(u(x))}{du} \cdot \frac{du(x)}{dx} = \cos u \cdot 3x^2$$

$$= (\cos 3x^2) \cdot 3x^2.$$

#.

§. 4.5: Integration by substitution

Ex: $\int (x^2+1)^2 (2x)\, dx$

$$\overset{u = \cancel{x}x^2+1}{\underset{\frac{du}{dx} = 2x}{=\!=}} \int u^2 \cdot du$$

$$= \frac{u^3}{3} + C = \frac{(x^2+1)^3}{3} + C$$

Ex: $\int 5 \cos 5x \, dx$

$$\overset{u = 5x}{\underset{\frac{du}{dx} = 5}{=}} \int \cos u \, du$$

$$= \sin u + C = \sin 5x + C.$$

Ex: $\int x (x^2+1)^2 \, dx$

$$\overset{u = x^2+1}{\underset{\frac{du}{dx} = 2x}{=}} \int (x^2+1)^2 \cdot \frac{1}{2} du$$

$$= \frac{1}{2} \int u^2 du$$

$$= \frac{1}{2} \cdot \frac{1}{3} u^3 + C$$

$$= \frac{u^3}{6} + C = \frac{1}{6} (x^2+1)^3 + C.$$

Ex: $\int \sqrt{2x-1}\, dx$

$$\overset{u=2x-1}{=\!=} \int u^{\frac{1}{2}} \cdot \frac{1}{2}\, du$$

$\frac{du}{dx} = 2$

$$= \frac{2}{3} \cdot \frac{1}{2}\, u^{\frac{3}{2}} + C$$

$$= \frac{1}{3}(2x-1)^{\frac{3}{2}} + C.$$

Ex: $\int x\sqrt{2x-1}\, dx$

$$\overset{u=2x-1}{=\!=} \int u^{\frac{1}{2}}\, x\, dx$$

$\frac{du}{dx} = 2$

$$= \int u^{\frac{1}{2}} \cdot \frac{1}{2}\, du \cdot \frac{u+1}{2}$$

$$= \frac{1}{4}\int (u+1)\, u^{\frac{1}{2}}\, du$$

$$= \frac{1}{4}\left[\int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du\right]$$

$$= \frac{1}{4}\left[\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} \cancel{+C}\right] + C$$

$$= \frac{1}{10}(2x-1)^{\frac{5}{2}} + \frac{1}{6}(2x-1)^{\frac{3}{2}} + C$$

#.

Ex: $\int \sin^2 3x \cos 3x \, dx$

$$\underset{\frac{du}{dx}=3\cos 3x}{\overset{u=\sin 3x}{=\!=}} \int u^2 \cdot \frac{1}{3} \, du$$

$$= \frac{1}{9} u^3 + C = \frac{1}{9} (\sin 3x)^3 + C.$$

Ex: $\int_0^1 x (x^2+1)^3 \, dx$

$x: 0 \to 1$

$u = x^2+1 : 1 \to 2$

$$\underset{\frac{du}{dx}=2x}{\overset{u=x^2+1}{=\!=}} \int_1^2 u^3 \frac{1}{2} \, du$$

$$= \frac{u^4}{8} \Big|_1^2 = \frac{16}{8} - \frac{1}{8} = \frac{15}{8}.$$

$$= \left(\frac{(x^2+1)^4}{8} \Big|_0^1 \right)$$

Ex: $\int_1^5 \frac{x}{\sqrt{2x-1}} \, dx$

$$\underset{\frac{du}{dx}=2}{\overset{u=2x-1}{=}} \int_1^9 u^{-\frac{1}{2}} \frac{1}{2} \, du \left(\frac{u+1}{2}\right)$$

$$= \frac{1}{4} \int_1^9 (u^{\frac{1}{2}} + u^{-\frac{1}{2}}) \, du$$

$$= \frac{1}{4} \left[\frac{2}{3} u^{\frac{3}{2}} + 2u^{\frac{1}{2}} \Big|_1^9 \right]$$

$$= \frac{1}{4} \left[\left(\frac{2}{3} \cdot 9^{\frac{3}{2}} + 2 \cdot 9^{\frac{1}{2}}\right) - \left(\frac{2}{3} + 2\right) \right]$$

$$= \frac{1}{4} \left[18 + 6 - \frac{2}{3} - 2 \right] = \frac{1}{4} \left(\frac{64}{3} \right) = \frac{16}{3} \#.$$

Thm: ~~fet~~ Let f be ~~integrall~~. integrable on $[-a, a]$

Then (1). if f is even function,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$f(x) = f(-x)$$

(2) if f is odd function,

$$\int_{-a}^{a} f(x)\,dx = 0. \qquad f(-x) = -f(x).$$