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Ch 3 Applications of Differentiation

~~Exe~~ ◎ Extrema of a function

absolute, relative

◎ How to find Extrema?

Definition: Let f be defined on an interval I containing c

1. $f(c)$ is the (absolute) minimum of f on I if

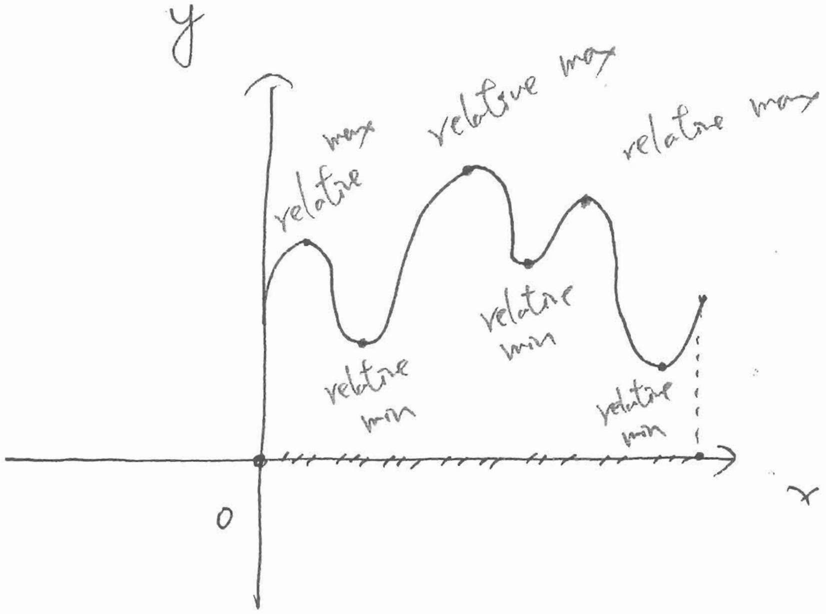
$$f(c) \le f(x) \quad \forall x \in I.$$

2. $f(c)$ is ~~the~~ (absolute) maximum of f on I if

$$f(c) \geq f(x) \quad \forall x \in I.$$

Definition: 1. if there is an open interval containing c on which $f(c)$ is a maximum (min.), then $f(c)$ is called a relative maximum (min) of f.

~~2. Def~~



<u>Theorem</u>: if f is cont. on a closed interval $[a, b]$,
(Existence) then f has both a mimum and a maximum on $[a. b]$

To find < absolute max (min)
relative max (min)

⊙ Extrema of a function (Extreme value) < max. / min. { relative (local) / absolute

⊙ Existence :

Theorem 3.1 : if f is cont on $[a, b]$, then f has both a min. and a max. on $[a, b]$

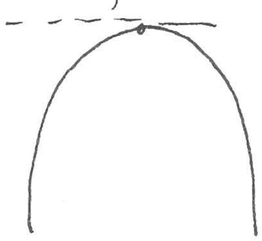
$\boxed{\text{⊙ How to find the } \left\{\begin{array}{l}\text{relative}\\ \text{absolute}\end{array}\right. \text{ extreme value}}$

To find the absolute extreme value :

Definition : Let f be defined at c.

if $f'(c) = 0$ or if f' is undefined at c, then c is a critical number of f

critical point



<u>Theorem</u>:

if f has a relative min. or relative max at $x=c$, then c is a critical number of f.

(f 在 c 產生相對極值 $\Rightarrow$ c 是一個臨界點)

$\not\Leftarrow$?

$\boxed{\text{Ex: } f(x) = x^3, \ c=0}$

<u>proof</u>: case (i) if f is not diff at $x=c$.

(o.k)

case (ii) if f is diff at $x=c$

(a) $f'(c)=0$ (o.k)

(b) $f'(c)>0 \Rightarrow f'(c) = \lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c} > 0.$

(c) $f'(c)<0.$

To find the absolute extreme value :

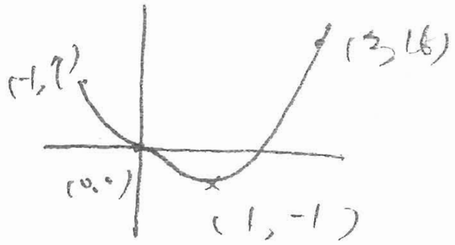
(i) Find critical number

(ii) Evaluate f at each critical number

(iii) Evaluate f at each endpoint of $[a,b]$.

(iv) The least of these value is the min,
the greatest ~~of~~ is max.

<u>Ex</u> : Find the extrema of $f(x) = 3x^4 - 4x^3$ on $[-1, 2]$.



<u>Sol</u> : $f(x) = 3x^4 - 4x^3$

$\Rightarrow f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$

$\Rightarrow$ critical numbers of f ~~is~~ are 0 and 1

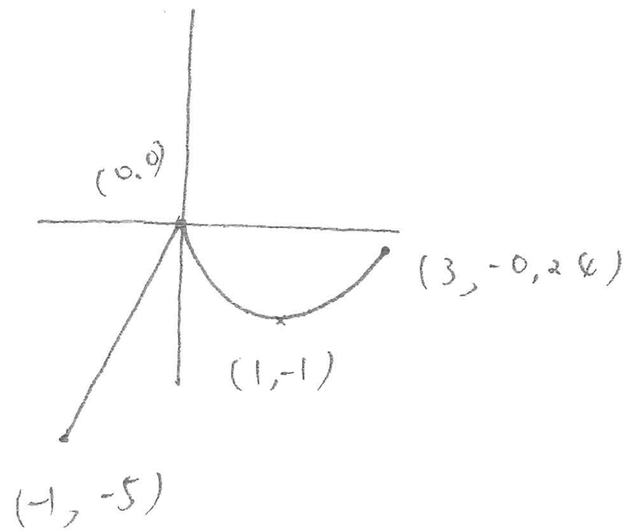
<u>Compute</u> : $f(0) = 0$, $f(1) = -1$

$f(-1) = 7$, $f(2) = 16$

$\therefore$ absolute max is 16 at 2.

absolute min is -1 at 1.

Ex: Find the extrema of $f(x) = 2x - 3x^{\frac{2}{3}}$ on $[-1, 3]$



sol: $f(x) = 2x - 3x^{\frac{2}{3}}$

$\Rightarrow f'(x) = 2 - 3 \cdot \frac{2}{3} x^{-\frac{1}{3}} = 2 - 2\frac{1}{x^{\frac{1}{3}}}$

$\Rightarrow$ critical numbers are $\underset{f'(1)=0}{\underline{1}}$ and $\underset{f'(0) \text{ is undefined}}{\underline{0}}$

Compute $f(0) = 0$, $f(1) = -1$

$f(-1) = -5$, $f(3) = 6 - 3\sqrt[3]{9} \approx -0.28$.

$\therefore$ absolute max is 0 at 0

absolute min is -5 at -1

Ex: Find the extrema of

$$f(x) = 2\sin x - \cos 2x \quad \text{on } [0, 2\pi]$$

sol:

$$f'(x) = 2\cos x + \sin 2x \cdot 2$$

$\therefore$ $f'(x) = 0 \Rightarrow \cos x + \sin 2x = 0$

$\Rightarrow \cos x + 2\sin x \cos x = 0$

$\Rightarrow \cos x (1 + 2\sin x) = 0$

$\Rightarrow \cos x = 0$ or $1 + 2\sin x = 0$

~~$\Rightarrow x = \frac{\pi}{2}, \frac{3}{2}\pi$~~

$\Rightarrow \underline{\cos x = 0}$ or $\sin x = -\frac{1}{2}$

$\Rightarrow \underline{x = \frac{\pi}{2}, \frac{3\pi}{2}}$ or $x = \frac{7}{6}\pi, \frac{11}{6}\pi$.

$\Rightarrow$ critical numbers are $\frac{\pi}{2}, \frac{3}{2}\pi, \frac{7}{6}\pi, \frac{11}{6}\pi$

Compute $f(\frac{\pi}{2}) = 3$, $f(\frac{3}{2}\pi) = -1$, $f(\frac{7}{6}\pi) = -\frac{3}{2}$

$f(\frac{11}{6}\pi) = -\frac{3}{2}$.

$f(0) = -1$, $f(2\pi) = -1$.

$\therefore$ absolute max is 3 at $\frac{\pi}{2}$.

absolute min is $-\frac{3}{2}$ at $\frac{7}{6}\pi, \frac{11}{6}\pi$.

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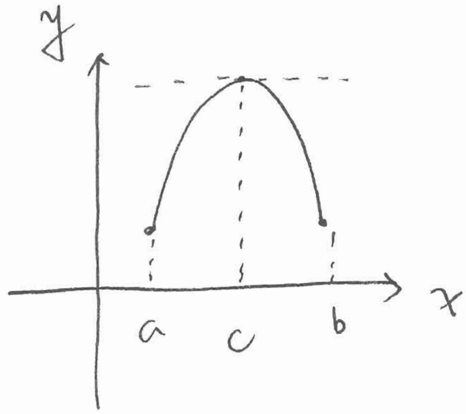
§ 3.2 Rolle's Theorem & Mean Value Theorem.

Theorem : (Rolle's Theorem)

Let f be cont. on $[a, b]$ and diff on (a, b). If

$$f(a) = f(b)$$

$\Rightarrow$ $\exists$ $c \in (a, b)$, s.t $f'(c) = 0$.



※ 極值產生在內點

Pf: Let $f(a) = d = f(b)$

case (i) $f(x) = d$ $\forall$ $x \in [a, b]$.

$\Rightarrow$ $f'(x) = 0$ $\forall$ $x \in (a, b)$

case (ii) $\exists$ $x \in (a, b)$, s.t $f(x) > d$

$\Rightarrow$ f attains its max at c, $c \in I$

$\because f(c) > d$ $\Rightarrow$ $\because$ c is not an endpoint

$\Rightarrow$ $f(c)$ is a relative max.

$\Rightarrow$ c is a critical number

$\Rightarrow$ $f'(c) = 0$.

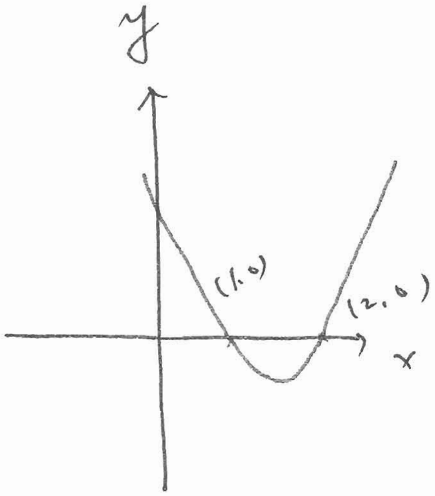
case (iii) $\exists\ x \in (a,b)$, s.th $f(x) < d$

Similarly to above argument.

Ex: Find the two x-intercepts of $f(x) = x^2 - 3x + 2$ and show that $f'(x) = 0$ at some point between the two intercepts.

Sol: (i) To find x-intercepts:

Solve $f(x) = x^2 - 3x + 2 = 0$



$\Rightarrow (x-2)(x-1) = 0$

$\Rightarrow x = 1$ or 2

(ii) Since f is cont on $[1,2]$ and diff on (a,b), and $f(1) = f(2) = 0$

by Rolle's Thm.

$\Rightarrow \exists\ c \in (1,2)$

s.t $f'(c) = 0$.

To find c :

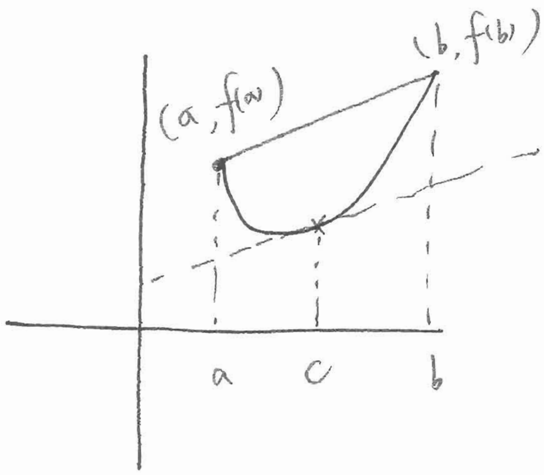
$$f'(x) = 2x - 3$$

$$\Rightarrow f'(x) = 0 \Rightarrow x = \frac{3}{2} \ \#.$$

Theorem (Mean Value Theorem).

If f is cont on $[a,b]$ and diff on (a,b), then $\exists\ c \in (a,b)$, s.th.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



(割線斜率 = 切線斜率)

secant line　　tangent line

(平均速度 = 瞬間速度)

Sol: The equation of the secant line

$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

$$\Rightarrow y = \frac{f(b)-f(a)}{b-a}(x-a) + f(a).$$

Let $g(x) = f(x) - y$.

$$= f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a).$$

$$\Rightarrow \begin{cases} g \text{ is cont on } [a,b]. \\ g \text{ is diff on } (a,b) \\ g(a) = g(b) = 0. \end{cases}$$

by Rolle's Thm.

$\Rightarrow \quad \exists\ c \in (a,b)$, s.th $g'(c) = 0$

ie $\quad g'(c) = f'(c) - \dfrac{f(b)-f(a)}{b-a} = 0.$

$$\Rightarrow \quad f'(c) = \frac{f(b)-f(a)}{b-a}$$

#.

$\boxed{\text{Mean value Thm}} \Leftrightarrow \boxed{\text{Rolle's Thm}}$

Ex: Given $f(x) = 5 - \frac{4}{x}$, find all value of c in $(1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}.$$

Sol:
$$\frac{f(4) - f(1)}{4 - 1} = \frac{4 - 1}{3} = 1$$

Since f satisfies the hypothesis of M.V.T

$\Rightarrow \exists\, c \in (1, 4)$ s.th $f'(c) = 1$.

To find c:

$$f'(x) = \frac{4}{x^2}$$

Solve $f'(x) = \frac{4}{x^2} = 1$

$\Rightarrow x = \pm 2$

$\Rightarrow c = 2$

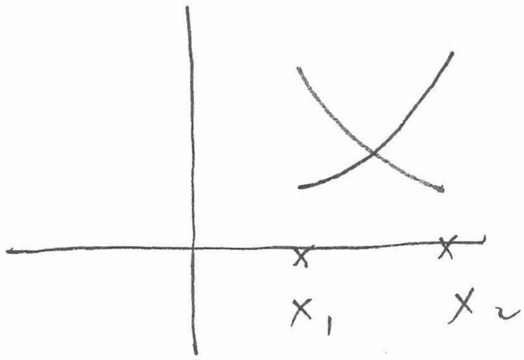
§ 3.3

$\boxed{\text{To find the relative extrema}}$:

Def: A function f is increasing (decreasing) on I

if $\forall\ x_1, x_2 \in I$ with $x_1 < x_2$

$$\Rightarrow f(x_1) < f(x_2)$$
$$(>)$$



Theorem: Let f be cont on $[a,b]$ and diff on (a,b). Then

(i) if $f'(x) > 0$ $\forall\ x \in (a,b)$

$\Rightarrow f$ is increasing on $[a,b]$.

(ii) if $f'(x) < 0$ $\forall\ x \in (a,b)$

$\Rightarrow f$ is decreasing on $[a,b]$.

(iii) if $f'(x) = 0$ $\forall\ x \in (a,b)$

$\Rightarrow f$ is constant on $[a,b]$.

pf: To prove (i)

Assume $f'(x) > 0 \quad \forall x \in (a, b)$

and $x_1 < x_2$

by M.V.T, $\exists\ c \in (a,b)$, s.th.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

$$\Rightarrow f(x_2) > f(x_1)$$

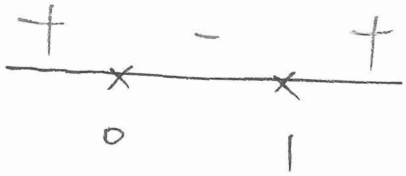
#

Ex: Find the open intervals on which $f(x) = x^3 - \frac{3}{2}x^2$ is increasing or decreasing.

Sol:

$$f'(x) = 3x^2 - \frac{3}{2}\cdot 2x = 3x^2 - 3x$$

$$= 3x(x-1)$$



$\therefore$ $\begin{cases} f \text{ is increasing on } (-\infty, 0) \text{ and } (1, \infty) \\ f \text{ is decreasing on } (0, 1) \end{cases}$ #

Theorem: (First Derivative Test)

$\begin{cases} f \text{ is cont on } I \text{ containing } c. \\ f \text{ is diff on } I \text{ (except possibly at } c) \\ c \text{ is a critical number.} \end{cases}$

Then

(i) if $f'(x)$ changes from negative to positive at c

$f'(x)<0$ $\quad$ $f'(x)>0$

c

$\Rightarrow$ $f(c)$ is relative min.

(ii) if $f'(x)$ changes from positive to negative at c

$f'(x)>0$ $\quad$ $f'(x)<0$

c

$\Rightarrow$ $f(c)$ is relative max.

(iii) if f' does not change sign at c

then f has relative extreme value at c.

Ex: Find the relative extrema of the function

$$f(x) = \frac{1}{2}x - \sin x \quad \text{in } (0, 2\pi).$$

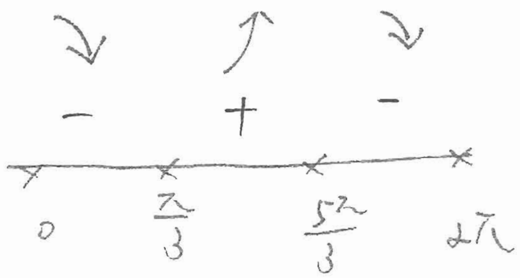
Sol:

$$f(x) = \frac{1}{2}x - \sin x$$

$$\Rightarrow f'(x) = \frac{1}{2} - \cos x$$

$$\Rightarrow f'(x) = 0 \iff x = \frac{\pi}{3}, \frac{5}{3}\pi$$

$\therefore$ critical number are $x = \frac{\pi}{3}, \frac{5}{3}\pi$.



by first-Derivative Test

$$\Rightarrow \begin{cases} f \text{ has relative min } f(\frac{\pi}{3}) = \frac{\pi}{6} - \frac{\sqrt{3}}{2} \text{ at } \frac{\pi}{3}. \\ \\ f \text{ has relative max } f(\frac{5}{3}\pi) = \frac{5}{6}\pi + \frac{\sqrt{3}}{2} \text{ at } \frac{5}{3}\pi \end{cases}$$

#.

Ex: Find the relative extrema of

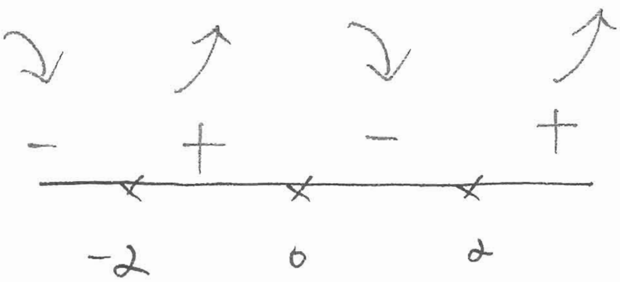
$$f(x) = (x^2-4)^{\frac{2}{3}}.$$

Sol: $$f(x) = (x^2-4)^{\frac{2}{3}}$$

$$\Rightarrow f'(x) = \frac{2}{3}(x^2-4)^{-\frac{1}{3}} \cdot 2x$$

$$= \frac{4x}{3\sqrt[3]{x^2-4}}$$

$\Rightarrow$ critical numbers are $x=0$ and $x=\pm 2$.



by first-derivative Test

$$\Rightarrow \begin{cases} f \text{ has relative min } f(-2)=0 \text{ at } x=-2. \\ f \text{ has relative max } f(0)=\sqrt[3]{16} \text{ at } x=0. \\ f \text{ has relative min } f(2)=0 \text{ at } x=2. \end{cases}$$

#

Ex: Find the relative extrema of

$$f(x) = \frac{x^4+1}{x^2}$$

Sol:

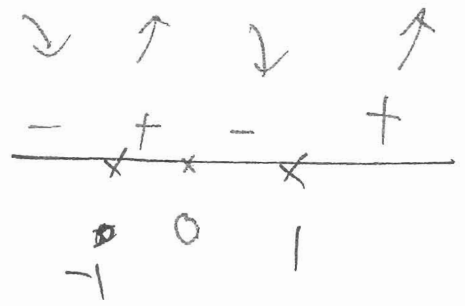
$$f(x) = \frac{x^4+1}{x^2}$$

$$\Rightarrow \quad f'(x) = \frac{4x^3 \cdot x^2 - (x^4+1)\cdot 2x}{x^4} = \frac{4x^5 - 2x^5 - 2x}{x^4}$$

$$= \frac{2x^5 - 2x}{x^4} = \frac{2(x^4-1)}{x^3}$$

$$= \frac{2(x^2+1)(x+1)(x-1)}{x^3}.$$

critical number are $x = \pm 1$.



by First-Derivative Test

$$\Rightarrow \begin{cases} f \text{ has relative min } f(-1) = 2 \text{ at } x=-1 \\ f \text{ has relative min } f(1) = 2 \text{ at } x = 1 \end{cases}$$

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To find the relative extrema.

(i) To find critical number

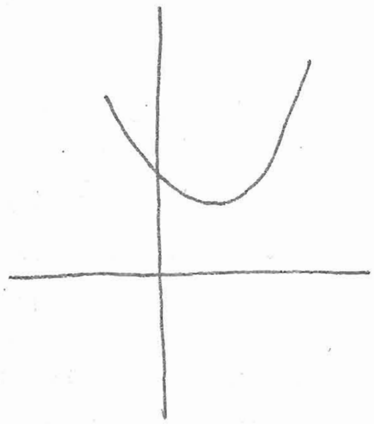
(ii) First-Derivative Test

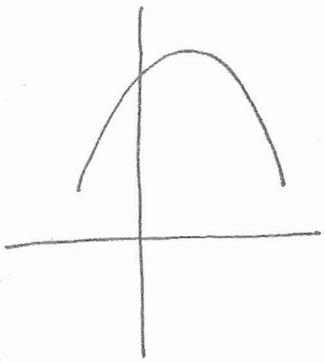
Second-Derivative Test

$\underset{c}{f'<0 \times f'>0}$: relative min , $\underset{c}{f'>0 \times f'<0}$: relative max.

Concavity:

Def. Let f be diff. on open interval I. The graph of f is concave upward on I if f' is increasing on I and concave downward on I if f' is decreasing on I





Thm:

(i) if $f''(x) > 0 \quad \forall\ x \in I$

$\Rightarrow$ the graph of f is concave up in I.

(ii) if $f''(x) < 0 \quad \forall\ x \in I$

$\Rightarrow$ the graph of f is concave down in I.

Ex: Determine the open intervals on which the graph of $f(x) = 6(x^2+3)^{-1}$ is concave up or down.

Sol:

$$f(x) = 6(x^2+3)^{-1}$$

$$\Rightarrow f'(x) = -6(x^2+3)^{-2} \cdot 2x = -12x(x^2+3)^{-2}$$

$$\Rightarrow f''(x) = \cancel{24x} \ -12(x^2+3)^{-2} + 24x(x^2+3)^{-3} \cdot 2x$$

$$= -12(x^2+3)^{-2} + 48x^2(x^2+3)^{-3}$$

$$= \frac{-12(x^2+3) + 48x^2}{(x^2+3)^3} = \frac{36x^2-36}{(x^2+3)^3}$$

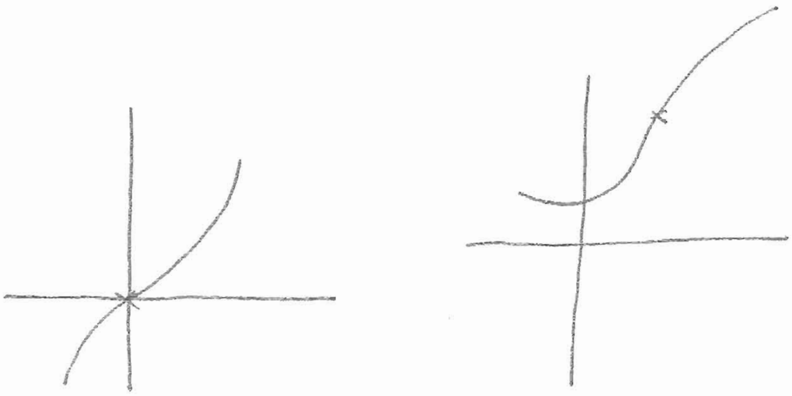
$$= \frac{36(x+1)(x-1)}{(x^2+3)^3}.$$

$f'' > 0 \qquad f'' < 0 \qquad f'' > 0$

$-1 \qquad\qquad 1$

$\therefore$ $\begin{cases} \text{the graph of } f \text{ is concave up on } (-\infty, -1) \text{ and } (1, \infty) \\ \text{the graph of } f \text{ is concave down on } (-1, 1) \end{cases}$

◎ <u>Points of inflection</u>



<u>Thm</u>: If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f'' is undefined at c.

(c is a point of inflection $\Rightarrow$ $f''(c) = 0$ or $f''(c)$ undefined
$\Leftarrow$)

Ex: Determine the points of inflection and discuss the concavity of the graph of

$$f(x) = x^4 - 4x^3$$

sol:

$$f(x) = x^4 - 4x^3$$

$$\Rightarrow f'(x) = 4x^3 - 12x^2$$

$$\Rightarrow f''(x) = 12x^2 - 24x = 12x(x-2)$$

$f''>0$		$f''<0$		$f''>0$
	0		2	

$\Rightarrow$
- Points of inflection are $x=0$ and $x=2$.
- the graph of f is concave up on $(-\infty, 0)$ and $(2, \infty)$
- " " " " is concave down on $(0, 2)$.

$f'<0 \quad f'>0 \quad f'>0 \quad f'<0$

$-1 \qquad 0 \qquad 1$

by first-Derivative Test

$\Rightarrow$ $(0,0)$ ~~$(x=0)$~~ is neither a relative max nor a relative ~~max~~ min.

In fact, $(0,0)$ is the point of inflection.

§ 3.5 Limits at infinity. 10/23

$$\begin{cases} \lim_{x\to\infty} \dfrac{3x^2}{x^2+1} = 3. \\ \lim_{x\to-\infty} \dfrac{3x^2}{x^2+1} = 3. \end{cases}$$

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Def: The line $\boxed{y = L}$ is a <u>horizontal asymptote</u> of the graph of f if

$$\boxed{\lim_{x\to-\infty} f(x) = L} \quad \text{or} \quad \boxed{\lim_{x\to\infty} f(x) = L.}$$

◎ <u>Vertical Asymptote</u> :

<u>horizontal asymptote</u> : $y = L$ <u>Check</u> : $\begin{cases} \lim_{x\to\infty} f(x) = L \\ \lim_{x\to-\infty} f(x) = L \end{cases}$

Vertical asymptote : $x = c$ Check : $\begin{cases} \lim_{x\to c^+} f(x) = \begin{cases} +\infty \\ -\infty \end{cases} \\ \lim_{x\to c^-} f(x) = \begin{cases} +\infty \\ -\infty \end{cases} \end{cases}$

Ex: $\lim_{x\to\infty}\left(5-\frac{2}{x^2}\right)=5$.

Ex: $\lim_{x\to\infty}\frac{2x-1}{x+1}=\lim_{x\to\infty}\frac{\frac{2x-1}{x}}{\frac{x+1}{x}}=\lim_{x\to\infty}\frac{2-\frac{1}{x}}{1+\frac{1}{x}}=2$.

Ex: $\lim_{x\to\infty}\frac{2x+5}{3x^2+1}=\lim_{x\to\infty}\frac{\frac{2}{x}+\frac{5}{x^2}}{3+\frac{1}{x^2}}=0$.

Ex: $\lim_{x\to\infty}\frac{2x^2+5}{3x^2+1}=\lim_{x\to\infty}\frac{2+\frac{5}{x^2}}{3+\frac{1}{x^2}}=\frac{2}{3}$.

Ex: $\lim_{x\to\infty}\frac{2x^3+5}{3x^2+1}=\lim_{x\to\infty}\frac{2x+\frac{5}{x^2}}{3+\frac{1}{x^2}}=\frac{\infty}{3}=\cancel{0}\infty$.

Ex: $\lim_{x\to\infty}\frac{3x-2}{\sqrt{2x^2+1}}$

$=\lim_{x\to\infty}\frac{3-\frac{2}{x}}{\sqrt{2+\frac{1}{x^2}}}=\frac{3}{\sqrt{2}}$.

Ex: $\lim_{x\to-\infty}\frac{3x-2}{\sqrt{2x^2+1}}$

$=\lim_{x\to-\infty}\frac{3-\frac{2}{x}}{-\sqrt{2+\frac{1}{x^2}}}=-\frac{3}{\sqrt{2}}$ #.

$\underline{x<0}$ $\quad x=-\sqrt{x^2}$

Ex : $\lim_{x\to\infty} \sin x$ does not exist.

Ex : $\lim_{x\to\infty} \frac{\sin x}{x}$

$\because \quad -1 \le \sin x \le 1$

$\Rightarrow \quad -\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

Since $\lim_{x\to-\infty} -\frac{1}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$

by Sandwich Thm (Squeeze) $\Rightarrow \lim_{x\to\infty} \frac{\sin x}{x} = 0$

#.

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§ A summary of Curve Sketching.

Ex: Analyze the graph of

$$f(x) = \frac{2(x^2-9)}{x^2-4}.$$

sol: (domain, range, intercepts, asymptotes,

critical number < relative max / relative min, points of inflection,

cont, diff, concavity)

(1) domain : $\mathbb{R} \setminus \pm\{2\}$.

(2) x-intercepts : $(-3, 0)$, $(3, 0)$

y-intercepts : $(0, \frac{18}{4}) = (0, \frac{9}{2})$.

(3) symmetry to y-axis

(4) $f(x) = \dfrac{2(x^2-9)}{x^2-4}$

$$\Rightarrow f'(x) = \frac{4x(x^2-4) - 4x(x^2-9)}{(x^2-4)^2} = \frac{20x}{(x^2-4)^2}$$

$$\Rightarrow f''(x) = \frac{20(x^2-4)^2 - 20x \cdot 2(x^2-4) \cdot 2x}{(x^2-4)^4} = \frac{20(x^2-4)(x^2-4-4x^2)}{(x^2-4)^4}$$

$$= \frac{-20(+3x^2+4)}{(x^2-4)^3}.$$

$\therefore$ critical number is $x=0$

point of inflection : None.

x	$-\infty < x < -2$	$x=-2$	$-2 < x < 0$	0	$0 < x < 2$	$x=2$	$2 < x < \infty$
~~$f(x)$~~	$\searrow$	undefined	$\searrow$	$\frac{9}{2}$	$\nearrow$	undefined	$\nearrow$
$f'(x)$	$-$	undefined	$-$	0	$+$	undefined	$+$
$f''(x)$	$-$	undefined	$+$	$+$	$+$	undefined	$-$
$f(x)$	$\searrow$ down	"	$\searrow$ up		$\nearrow$ up		$\nearrow$ down

(5) asymptote :

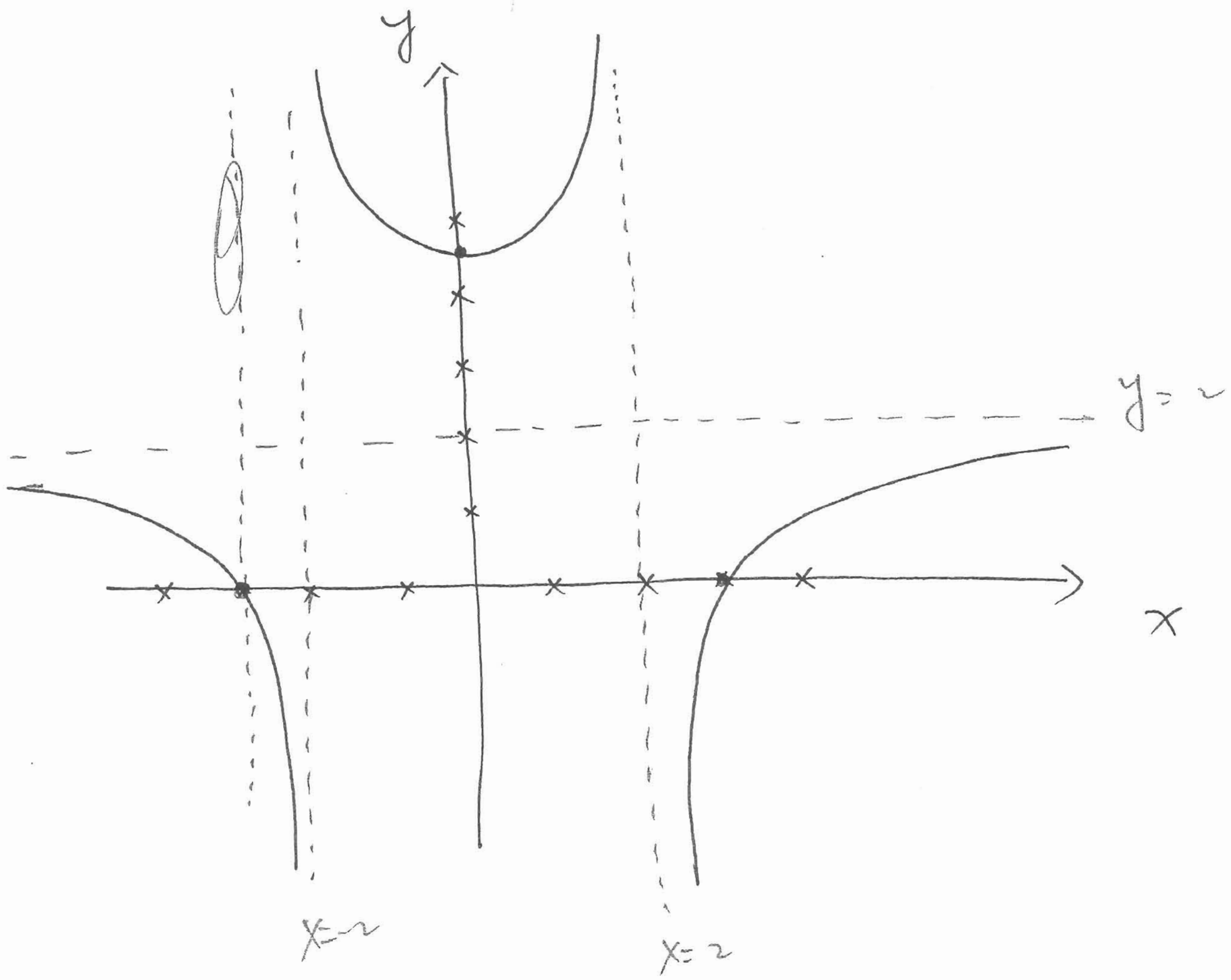
vertical :

$$\begin{cases} \lim\limits_{x\to 2^+} \dfrac{2(x^2-9)}{x^2-4} = -\infty . \\ \lim\limits_{x\to 2^-} \dfrac{2(x^2-9)}{x^2-4} = \infty \\ \lim\limits_{x\to (-2)^+} \dfrac{2(x^2-9)}{x^2-4} = +\infty \\ \lim\limits_{x\to (-2)^-} \dfrac{2(x^2-9)}{x^2-4} = -\infty . \end{cases}$$

$\Rightarrow$ vertical asymptotes : $x=2$, $x=-2$.

horijontal : $\begin{cases} \lim\limits_{x\to\infty} \dfrac{2(x^2-9)}{x^2-x} = 2. \\ \lim\limits_{x\to-\infty} \dfrac{2(x^2-9)}{x^2-x} = 2 \end{cases}$

$\Rightarrow$ horijontal asymptote : $y = 2$.



Ex: Analyze the graph of

$$f(x) = \frac{x}{\sqrt{x^2+2}}.$$

Sol: (1) domain : $\mathbb{R}$.

(2) $\begin{cases} x\text{- intercepts} : (0.0) \\ y\text{- intercepts} : (0.0) \end{cases}$

(3) symmetry to origin $\underline{(x, y)\ (-x, -y)}$

(4) $f(x) = \frac{x}{\sqrt{x^2+2}} = \frac{x}{(x^2+2)^{\frac{1}{2}}}$

$\Rightarrow f'(x) = \frac{(x^2+2)^{\frac{1}{2}} - x \cdot \frac{1}{2}(x^2+2)^{-\frac{1}{2}} \cdot 2x}{(x^2+2)} = \frac{(x^2+2)^{\frac{1}{2}} - x^2(x^2+2)^{-\frac{1}{2}}}{(x^2+2)}$

$= \frac{x^2+2-x^2 \cancel{(x^2+2)}}{(x^2+2)^{\frac{3}{2}}} = \frac{2}{(x^2+2)^{\frac{3}{2}}}$

$\Rightarrow f''(x) = \frac{}{\cancel{(x^2+2)^3}}\ 2 \cdot \left(-\frac{3}{2}\right)(x^2+2)^{-\frac{5}{2}} \cdot 2x$

$= \frac{-6x}{(x^2+2)^{\frac{5}{2}}}$.

Ex: Analyze the graph of

$$f(x) = \frac{x}{\sqrt{x^2+2}}.$$

Sol:

(1) domain : $\mathbb{R}$.

(2) $\begin{cases} x\text{- intercepts} : (0.0) \\ y\text{- intercepts} : (0.0) \end{cases}$

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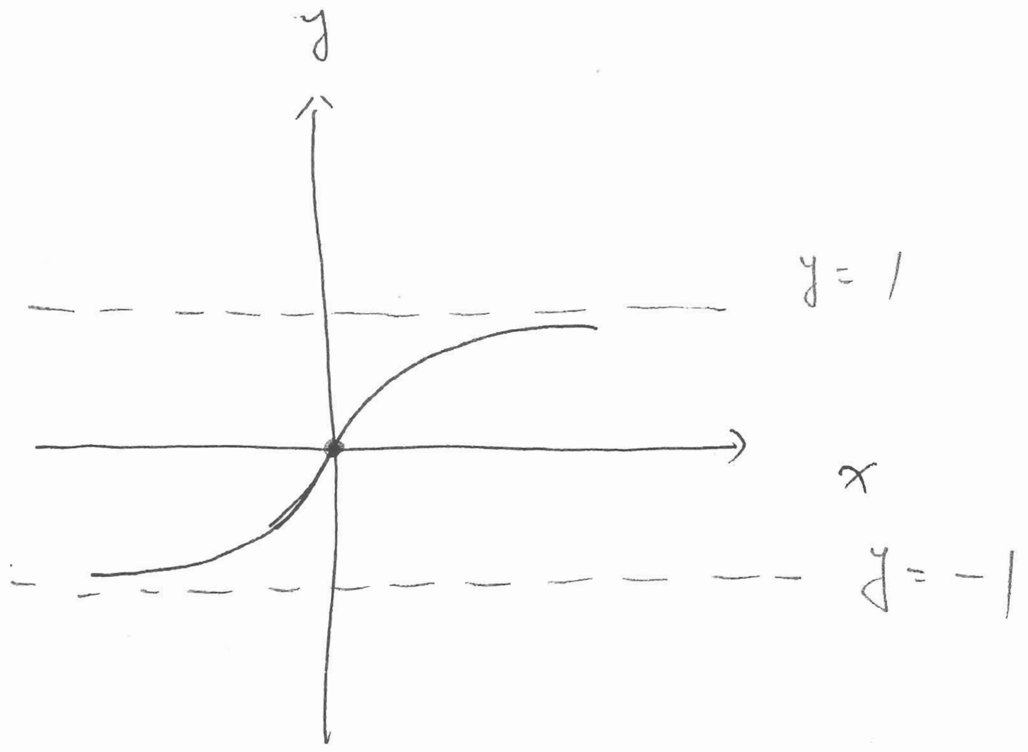
$$= \frac{x^2+2-x^2\cancel{(x^2+2)}}{(x^2+2)^{\frac{3}{2}}} = \frac{2}{(x^2+2)^{\frac{3}{2}}}$$

$$\Rightarrow f''(x) = \frac{\quad}{\cancel{(x^2+2)^3}}\ 2 \cdot \left(-\frac{3}{2}\right)(x^2+2)^{-\frac{5}{2}} \cdot 2x$$

$$= \frac{-6x}{(x^2+2)^{\frac{5}{2}}}.$$



Ex: Analyze the graph of

$$f(x) = x^4 - 12x^3 + 48x^2 - 64x.$$

Sol: (1) domain : $\mathbb{R}$.

(2) x-intercepts : $x^4 - 12x^3 + 48x^2 - 64x = 0$

$\Rightarrow x(x^3 - 12x^2 + 48x - 64) = 0$

$\Rightarrow x(x-4)^3 = 0$

$\therefore$ x-intercepts : $(0,0)$ $(4,0)$

y-intercepts : $(0,0)$.

(3) $f(x) = x^4 - 12x^3 + 48x^2 - 64x$

$\Rightarrow f'(x) = 4x^3 - 36x^2 + 96x - 64$

$\Rightarrow f''(x) = 12x^2 - 72x + 96$

$\therefore f'(x) = 0 \Rightarrow x^3 - 9x^2 + 24x - 16 = 0$

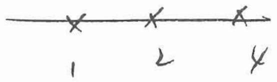
$\Rightarrow (x-1)(x^2 - 8x + 16) = 0$

$\Rightarrow (x-1)(x-4)^2 = 0.$

$f''(x) = 0 \Rightarrow 12x^2 - 72x + 96 = 0$

$\Rightarrow x^2 - 6x + 8 = 0$

$\Rightarrow (x-4)(x-2) = 0.$



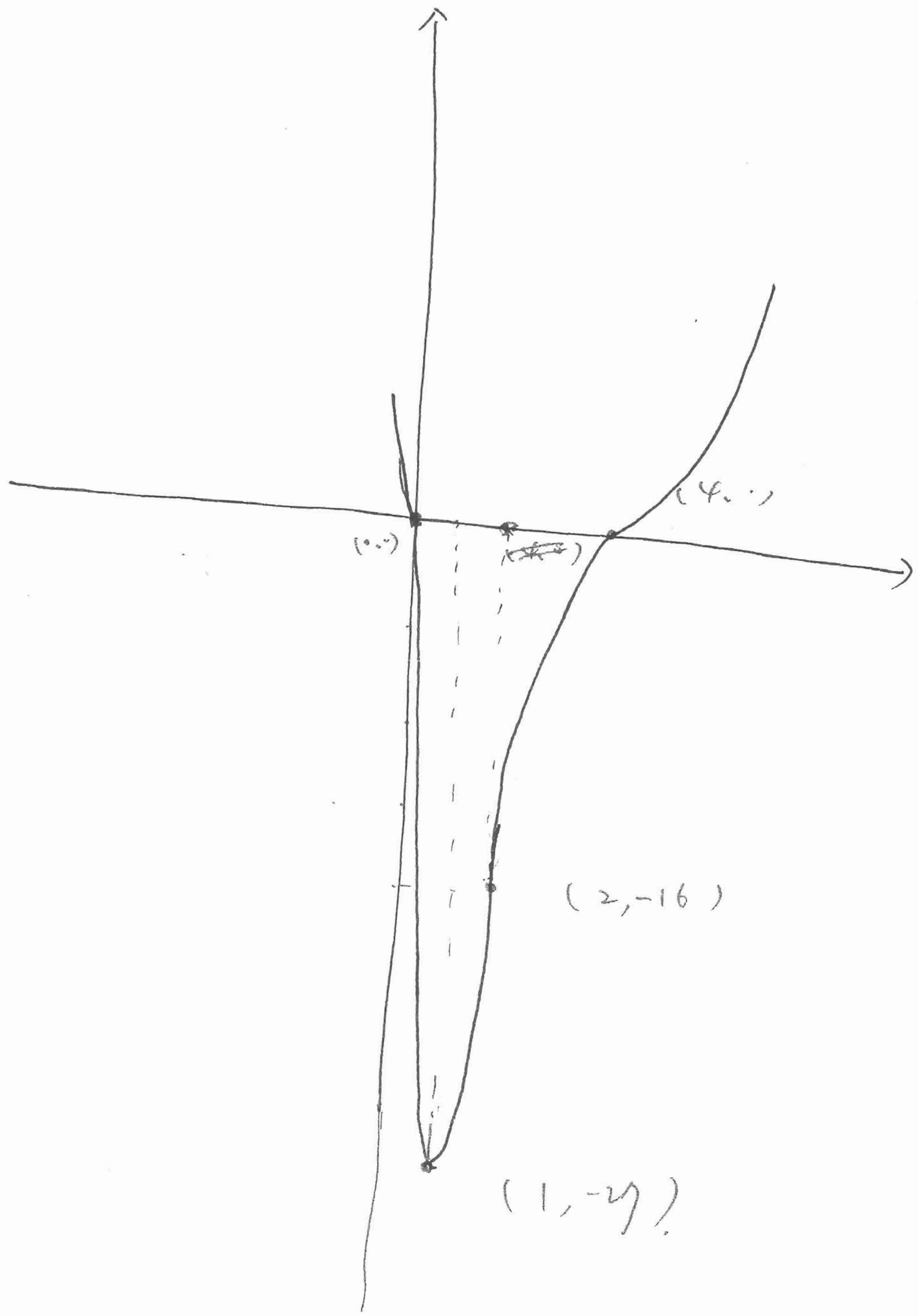
$\therefore$ critical numbers are $\underline{x=1}$, $\underline{x=4}$.

x	$-\infty<x<1$	$x=1$	$1<x<2$	$x=2$	$2<x<4$	$x=4$	$4<x<\infty$
$f'(x)$	$-$	0	$+$	$+$	$+$	0	$+$
$f''(x)$	$+$	$+$	$+$	0	$-$	0	$+$
$f(x)$	↓, up	-27 relative min.	↑, up	$-\cancel{0}16$ point of inflation	↑, ~~down~~ down	0 point of inflection	↑, ~~down~~ up

(4) asymptote :

vertical : none

horizontal : none .



Ex: Analyze the graph of

$$f(x) = \frac{\cos x}{1+\sin x}.$$

Sol:

(1) domain: $\mathbb{R} \setminus \{x ;\ x = \frac{3}{2}\pi + 2n\pi,\ n \in \mathbb{Z}\}$

(2) period $= 2\pi$ $\therefore$ (∵ 僅須考慮 $(-\frac{\pi}{2}, \frac{3}{2}\pi)$)

(3) x-intercepts: $(\frac{\pi}{2}, 0)$

y-intercepts: $(0, 1)$

(4)

$$f(x) = \frac{\cos x}{1+\sin x}$$

$$\Rightarrow f'(x) = \frac{-\sin x(1+\sin x) - \cos x(\cos x)}{(1+\sin x)^2} = \frac{-\sin x - 1}{(1+\sin x)^2}$$

$$= \frac{-1}{1+\sin x}$$

$$\Rightarrow f''(x) = (-1)(1+\sin x)^{-2} \cdot \cos x = \frac{-\cos x}{(1+\sin x)^2}$$

$\therefore$ ~~$f'(x) = 0 \Rightarrow \frac{\ }{1+s}$~~

$$f''(x) = 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}.$$

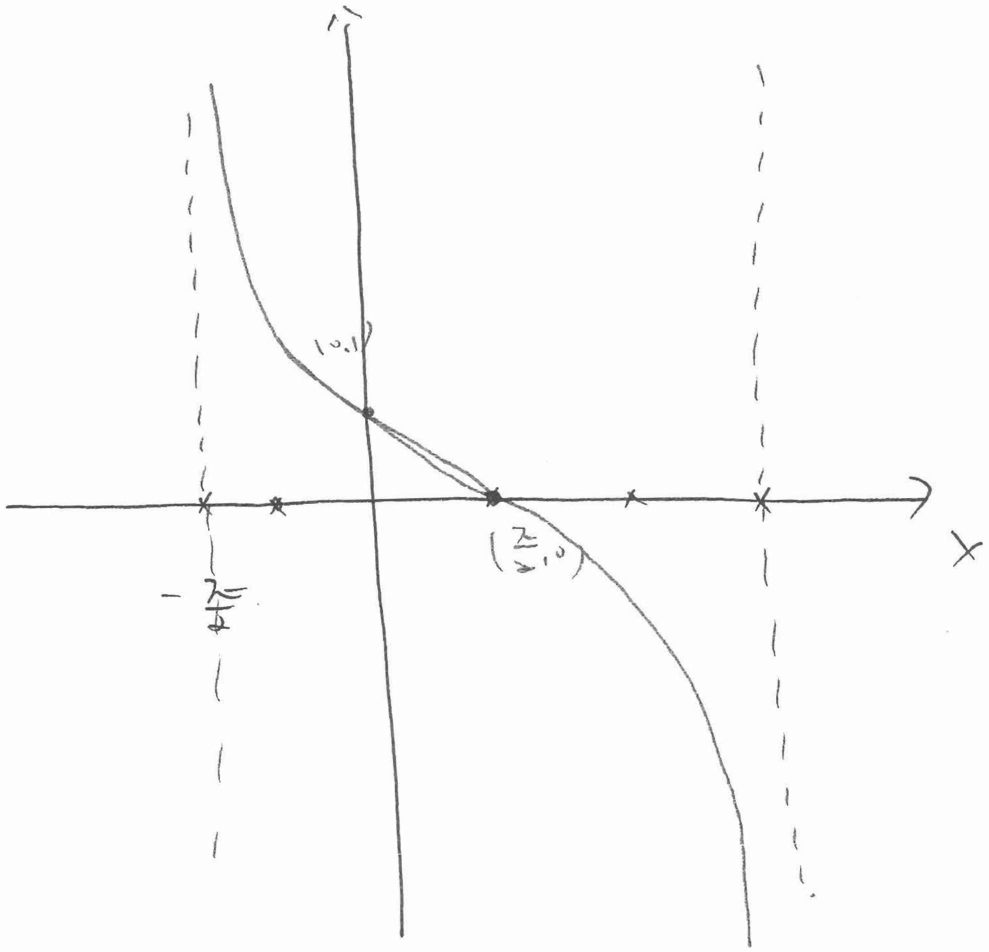
(5)

x	$-\frac{\pi}{2}$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$	$x = \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3}{2}\pi$
$f'(x)$		$-$	$-\frac{1}{2}$	$-$	
$f''(x)$		$+$	0	$-$	
$f(x)$		↓, up	0 Point of inflection	↓, down	

(5) asymptote :

vertical : $x = -\frac{\pi}{2}$, $x = \frac{3}{2}\pi$

horizontal : none.



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$$\begin{tikzcd} & \boxed{\text{function}} \arrow[dl] \arrow[d] \arrow[dr] & \\ \boxed{\text{limit}} & \boxed{\text{diff.}} & \boxed{\text{integral}} \end{tikzcd}$$

$$\begin{cases} f: [a,b] \to \mathbb{R} \\ f: E \subset \mathbb{R}^n \to \mathbb{R}. \\ f: [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}. \end{cases}$$

- ~~To f~~ ~~Ca.~~ Compute
  Calculate the limit
  ~~To find the~~
- To find the derivative
- Graph
- extrema and relative extreme
- ~~limit exists~~ 極限存在

  連續函數
  cont. fn

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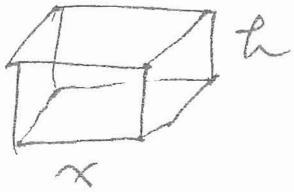
[illegible]

Ex: A manufacturer wants to design an open box having a square base and a surface area of 108 square inches, what dimensions will product a box with maximum volume?

sol: Volume : $V = x^2 h$



Surface ~~of~~ area : $x^2 + 4xh$

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已知 : $x^2 + 4xh = 108$

$$\Rightarrow h = \frac{108 - x^2}{4x}$$

$$\therefore V = x^2 \cdot \frac{108 - x^2}{4x}$$

$$= \frac{108x - x^3}{4} = 27x - \frac{x^3}{4}$$

◎ $0 \leq x \leq \sqrt{108}$ feasible domain

$$\frac{dV}{dx} = 27 - \frac{3}{4}x^2$$

$\Rightarrow$ ~~critical~~ $\frac{dV}{dx} = 0 \Rightarrow 3x^2 = 108 \Rightarrow x = \pm 6$

$\Rightarrow$ critical number $x = \pm 6$.

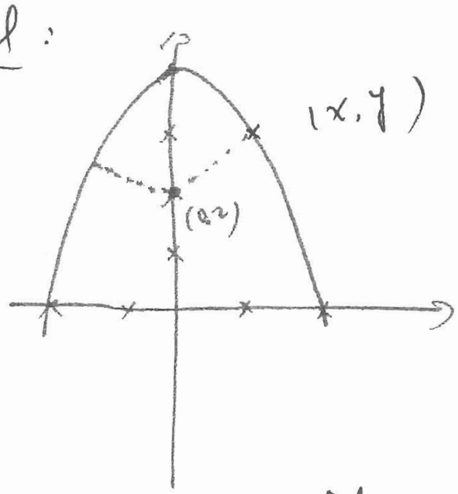
$$\begin{cases} V(0) = 0 \\ V(6) = 108 \\ \cancel{V(-6)} \\ V(\sqrt{108}) = 0. \end{cases}$$

$\therefore$ V is max. when $x = 6$.

and the dimensions of the box are $6 \times 6 \times 3$ inches.

Ex 2: Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$.

Sol:



$$d = \sqrt{(x-0)^2 + (y-2)^2}$$

$$\overset{y=4-x^2}{=} \sqrt{4 - y + y^2 - 4y + 4}$$

$$= \sqrt{y^2 - 5y + 8}$$

Set $d(y) = \sqrt{y^2 - 5y + 8}$ for $y \in (-\infty, 4]$.

$$d'(y) = \frac{1}{2}(y^2 - 5y + 8)^{-\frac{1}{2}} \cdot (2y - 5)$$

$$= \frac{1}{2} \frac{2y - 5}{\sqrt{y^2 - 5y + 8}}$$

$d'(y) = 0 \Rightarrow y = \frac{5}{2}$

~~$d'(y)$ does not exist $\Rightarrow y^2 - 5y + 8 = 0$~~

~~$\Rightarrow y = \frac{5 \pm \sqrt{\ }}{2}$~~

check : $\begin{cases} d(4) = \sqrt{4} = 2 . \\ d(\frac{5}{2}) = \sqrt{\frac{25}{4} - \frac{25}{2} + 8} = \sqrt{\frac{-25+32}{4}} = \sqrt{\frac{7}{4}} = \frac{\sqrt{7}}{2} \end{cases}$

$\therefore$ the min. distance is $\frac{\sqrt{7}}{2}$

and the points are $(\frac{\sqrt{3}}{2}, \frac{5}{2})$, $(-\frac{\sqrt{3}}{2}, \frac{5}{2})$

#

§. 3.9 Differentials

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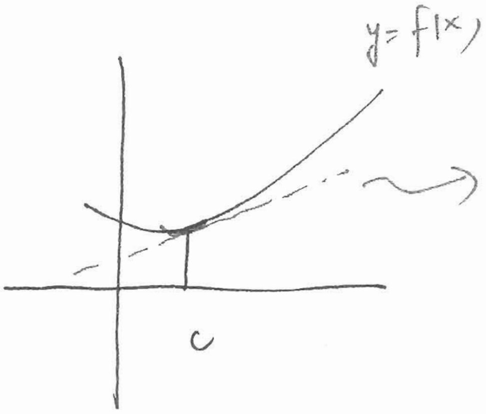


$$y - f(c) = f'(c)(x-c)$$

$$\Rightarrow \underline{y = f(c) + f'(c)(x-c)} \quad \text{tangent line}$$

Ex: Find the tangent line approximation of

$$f(x) = 1 + \sin x$$

at the (0.1). Then use a table to compare the y-values of the linear function with ~~those~~ of $f(x)$ in an open interval containing $x=0$.

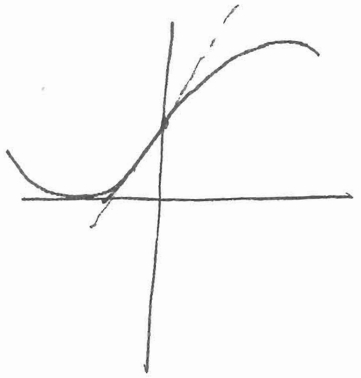
sol: $f(x) = 1 + \sin x \Rightarrow f'(x) = \cos x$

$f'(0) = \cos 0 = 1$

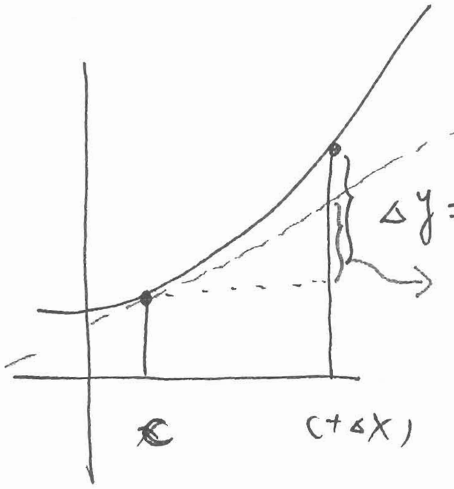
$\therefore$ tangent line: $y - f(0) = 1\,(x-0)$

$\Rightarrow y - 1 = x \Rightarrow y = 1 + x$.

$$\begin{cases} f(x) = 1 + \sin x \\ y = 1 + x \end{cases}$$



x	-0.5	-0.1	-0.01	0	0.01	0.1	0.5
$f(x)$	$+0.521$	0.9002	0.990002	1	1.0099998	1.0998	1.479
$y = 1 + x$	0.5	0.9	0.99	1	1.01	1.1	1.5



$y = f(c) + f'(c)(x-c) \Rightarrow y - f(c) = f'(c)(x-c)$

$\Delta y = f(c+\Delta x) - f(c)$

$f'(c)\Delta x$

$$\boxed{\Delta y \approx f'(c)\,\Delta x}$$

Def. The differential of x (denoted by x) is any non-zero real number. The differential of y (denoted by y) is $\boxed{dy = f'(x)\,dx}$

$$\cancel{dy} \quad \Delta y \approx dy$$

Ex: Let $y = x^2$. Find dy when $x = 1$ and $dx = 0.01$. Compare this value with Δy for $x = 1$ and $\Delta x = 0.01$.

Sol:

$$y = x^2 = f(x)$$

$$\Rightarrow \quad y' = 2x$$

$$\therefore \quad dy = 2x\,dx \xRightarrow[dx=0.01]{x=1} dy = 2 \times 0.01 = \underline{\underline{0.02}}$$

$$\begin{aligned} \Delta y &= f(x+\Delta x) - f(x) \\ &= f(1+0.01) - f(1) \\ &= (1.01)^2 - 1 \\ &= \underline{\underline{0.0201}} \end{aligned}$$

Ex: Find the differential of

$$y = f(x) = \sin 3x$$

Sol:

$$f(x) = \sin 3x$$

$$\Rightarrow f'(x) = 3\cos 3x$$

$$\therefore\ dy = f'(x)\,dx = 3\cos 3x\,dx$$

Ex: Use differentials to approximate $\sqrt{16.5}$

Sol: Let $f(x) = \sqrt{x}$, we write

$$f(x+\Delta x) - f(x) \approx dy = f'(x)\,dx$$
$$= \frac{1}{2} x^{-\frac{1}{2}}\,dx$$

Take $x = 16$, $\Delta x = 0.5$

$$\Rightarrow f(16.5) \approx f(16) + \cancel{\frac{1}{2} x^{-\frac{1}{2}}\,dx}\ \frac{1}{2}\frac{1}{\sqrt{16}}\ 0.5$$

$$\therefore\ \sqrt{16.5} \approx 4 + \frac{1}{2}\cdot\frac{1}{4}\times\frac{1}{2}$$

$$= 4 + \frac{1}{16}$$

#.

Ex: Use differentials to approximate $\sqrt[3]{28}$

Sol: Let $f(x) = \sqrt[3]{x}$

$$\Rightarrow f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$$

We write

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x$$

Take $x = 27$, $\Delta x = 1$

$$\sqrt[3]{28} \approx 3 + \frac{1}{3} \frac{1}{\sqrt[3]{3^{3}}^{2}} \cdot 1$$

$$= 3 + \frac{1}{3} \cdot \frac{1}{9} = 3 + \frac{1}{27}$$

#.