Question : $f_x = 2xy$, ~~✱~~ , $f(x,y) = ?$

Sol : $f(x,y) = \int f_x(x,y)\, d\cancel{y}x$
$= \int 2xy\, dx$
$= 2y \int x\, dx$
$= 2y \cdot \left(\frac{x^2}{2} + C\cancel{\ }(y)\right)$

Moreover $\int_1^{2y} \cancel{\ } 2xy\, d\cancel{y}x = 2y \int_1^{2y} x\, dy = 2y \cdot \frac{x}{2} \Big|_{x=1}^{x=2y}$

$= y \cdot (2y) - y \cdot 1 = 2y^2 - y$

iterated
~~iterated~~ integral

Ex $\int_1^2 \int_1^x (2x^2 y^{-2} + 2y)\, dy\, dx$

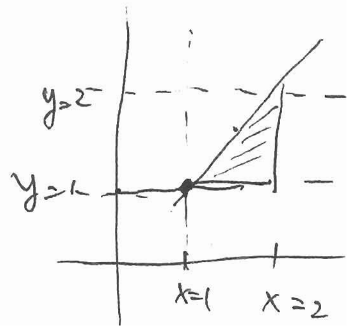
$= \int_1^2 2x^2 \frac{y^{-\cancel{1}}}{-\cancel{1}} + 2 \cdot \frac{y^2}{2} \Big|_{\cancel{y}1}^{x} dx$

$= \int_1^2 \left(2x^2 \cdot \frac{x^{-\cancel{1}}}{-\cancel{1}} + x^2\right) - \left(2x^2 \frac{1}{-\cancel{1}} + 1^2\right) dx$

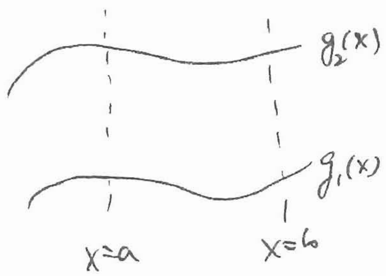
$= \int_1^2 (3x^2 - 2x - 1)\, dx$

$= x^3 - x^2 - x \Big|_1^2$

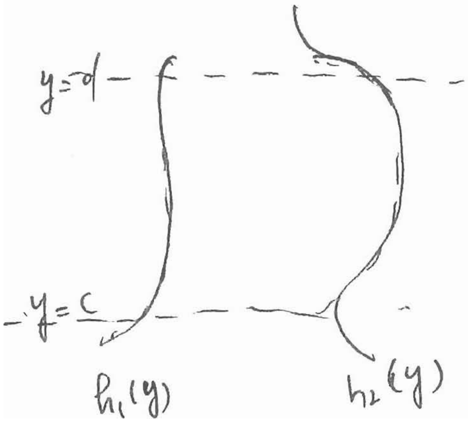
$= 3$



$R = \{ (x,y) \cancel{:}. 1 \le x < 2,\ 1 < y < x \}$. region of integration. R

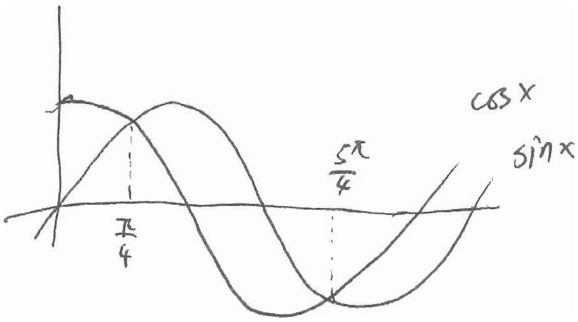


$$\int_a^b \int_{g_1(x)}^{g_2(x)} dy \quad dx$$

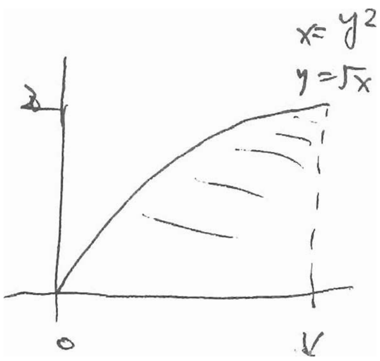


$$\int_c^d \int_{h_1(y)}^{h_2(y)} dx \quad dy.$$

Ex



$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \int_{\cos x}^{\sin x} dy\, dx = 2\sqrt{2}$$



$R = \; 0 \le y \le 2$
$\quad\; y^2 \le x \le 4$

or

$R: \; 0 \le x \le 4$
$\quad\; 0 \le y \le \sqrt{x}$

$$\int_0^2 \int_{y^2}^{4} dx \quad dy$$

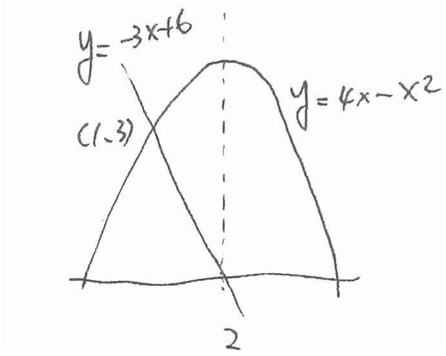
$$\parallel$$

$$\int_0^4 \int_0^{\sqrt{x}} dy\, dx$$

$$\parallel$$

$$\frac{16}{3}$$

Ex



$$\int_1^2\int_{-3x+6}^{4} \quad + \int_2^4\int_0^{4x-x^2}$$

$R:\quad 1\le x\le 2\ ,\quad -3x+6\le y\le -3x+6$

$\qquad 2\le x\le 4\ ,\quad 0\le y\le 4x^2-x^2$

14.2

R region $\subseteq \mathbb{R}^3$

Def $f(x,y) \geq 0 \quad \forall (x,y) \in R.$

$$\text{Val} = \lim_{\|\Delta\| \to 0} \sum f(x_i, y_i)\, \Delta A_i$$

$$= \iint_R f(x,y)\, dA \qquad \underline{\text{double integral of } f \text{ over } R}$$

$f \qquad \underline{\text{integrable}}$ over R.

Properties $\displaystyle \iint_R (cf \pm g)\, dA = c \iint_R f\, dA \pm \iint_R g\, dA$

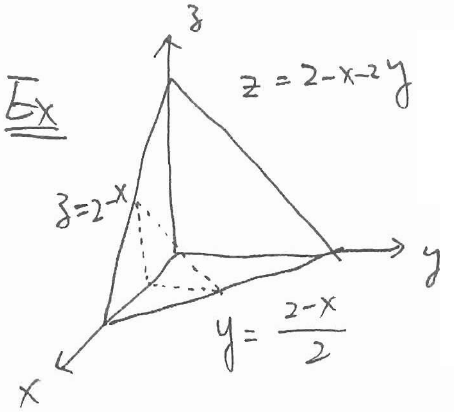
$$f(x,y) \geq g(x,y) \quad \forall (x,y) \in R \Rightarrow \iint_R f(x,y)\, dA > \iint_R g(x,y)\, dA$$

$$R = R_1 \cup R_2,\ R_1 \cap R_2 = \emptyset \Rightarrow \iint_R f\, dA = \iint_{R_1} f\, dA + \iint_{R_2} f\, dA$$

<u>How to calculate the double integral</u>

sliced . using iterated integral

Ex



$$A(x) = \frac{1}{2} \cdot \left(\frac{2-x}{2}\right) \cdot (2-x) = \frac{(2-x)^2}{4}$$

$$\text{Vol} = \int_a^b A(x)\, dx = \frac{2}{3}$$

Now $\displaystyle A(x) = \int_0^{\frac{2-x}{2}} (2 - x - 2y)\, dy$

$$\iint_R f(x,y,z)\, dA = \int_0^2 A(x)\, dx = \int_0^2 \int_0^{\frac{2-x}{2}} (2-x-2y)\, dy\,dx$$

Thm: Fubini's theorm

f continuous on R

(1) $R = \{ (x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x) \}$

$g_1(x), g_2(x)$ continuous on $[a,b]$

$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

(2) $R = \{ (x,y) \ ; \ c \le y \le d,\ h_1(y) \le x \le h_2(y) \}$

h_1, h_2 conti on $[a,b]$

$$\iint_R f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

Note: The iterated integral and the double integral have their own definition respectively.

But Fubini's them say that under the condition ~~that~~ of the continuity of f they give the same value.

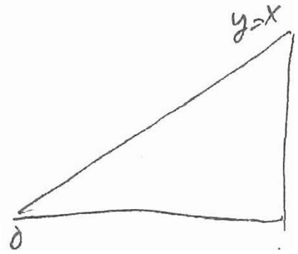
Ex $\iint_R (1 - \frac{1}{2}x^2 - \frac{1}{2}y^2)\, dA$ $\qquad R = \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{matrix}$

$= \frac{2}{3}$

Ex $f(x,y) = e^{-x^2}$ $R = \{(x,y) ; 0 \le x \le 1, \ 0 \le y \le x\}$

Sol:

$$\iint_R f(x,y)\, dA = \int_0^1\int_0^x e^{-x^2}\, dy\, dx = \frac{e-1}{2e}$$



Since $R = \{(x,y) : 0 \le y \le 1, \ y \le x \le 1\}$

Ex Vol of R bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below the plane $z = 1 - y$.

sol

$$Vol = \int_0^1 \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} \left((1-x^2-y^2) - (1-y)\right) dx\, dy.$$

$$= \frac{4}{3}\int_0^1 (y-y^2)^{\frac{3}{2}}\, dy$$

$2y - 1 = \sin\theta$

$$= \frac{1}{6}\int_{-\pi/2}^{\pi/2} \frac{\cos^4\theta}{2}\, d\theta$$

$$\Delta A = (r_i \Delta\theta)(\Delta r_i)$$

$$\iint_R f(r,\theta)\,dA \approx \sum_{i=1}^{n} f(r_i, \theta_i)\,(r_i \Delta\theta_i)\cdot(\Delta r_i)$$

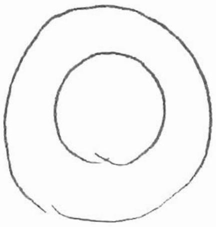
therefore

$$\iint_R f(r,\theta)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\; r\,dr\,d\theta$$

if $A = \{ (r,\theta) ;\ \alpha \le \theta \le \beta \ \ 0 \le g_1(\theta) \le r \le g_2(\theta) \}$

g_1, g_2 conti on$[\alpha,\beta]$, $0 \le \beta - \alpha \le 2\pi$

Ex



$R = \{ (x,y) ;\ 1 \le x^2+y^2 \le 5 \}$, $f(x,y) = x^2+y$.

$$\iint_R f(x,y)\,dA = ?$$

$$\Rightarrow \iint_R f(x,y)\,dA$$

$$= \int_0^{2\pi}\int_1^{\sqrt{5}} (r\cos\theta)^2 + r\sin\theta \quad r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(6\cos^2\theta + \frac{5\sqrt{5}-1}{3}\sin\theta\right) d\theta$$

$$= 3\theta + \frac{3\sin 2\theta}{2} - \frac{5\sqrt{5}-1}{3}\cos\theta \Big|_0^{2\pi}$$

Ex

Find the volume of the solid region bounded above by the hemisphere

$$z = \sqrt{16-x^2-y^2}$$

and below the circular region R given by

$$x^2+y^2 \le 4$$

pf

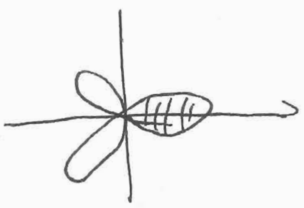
$$V = \iint_R f(x,y)\, dA$$

$$R = \{(x,y) \mid \cancel{\ldots}\ x^2+y^2 \le 4\}$$

$$= \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$$

$$V = \int_0^{2\pi}\int_0^2 \sqrt{16-r^2}\ r\,dr\,d\theta = \frac{16\pi}{3}(8-3\sqrt{3})$$

Ex $r = 3\cos 3\theta$



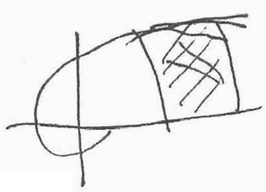
$$\frac{1}{3}A = \iint dA = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_0^{3\cos 3\theta} r\,dr\,d\theta$$

$$= \frac{9}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 3\theta\, d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{1}{6}\sin 6\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

between

Ex $r = \frac{\pi}{3\theta}$, ~~$r=1$, $r=2$~~ ~~between~~ $r=1$ & $r=2$.



$$\iint_R dA = \int_1^2\int_0^{\frac{\pi}{3r}} r\,d\theta\,dr = \frac{\pi}{3}$$

$$R = \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{3r}\}$$

14-5

$z = f(x,y)$

$\Delta T_i = \Delta S_i$

$\vec{u} = \Delta x_i \vec{i} + f_x(x_i, y_i)\,\Delta x_i \vec{k}$

$\vec{v} = \Delta y \vec{j} + f_y(x_i, y_i)\,\Delta y_i \vec{k}$

$$\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ \Delta x & 0 & f_x \Delta x \\ 0 & \Delta y & f_y \Delta y \end{vmatrix}$$

$$= -f_x \Delta y\, \Delta x \Delta y\, \vec{i} - f_y \Delta x \Delta y\, \vec{j} + \Delta x \Delta y\, \vec{k}$$

$$= (-f_x \vec{i} - f_y \vec{j} + \vec{k})\, \overbrace{\Delta x_i \Delta y_i}^{\Delta A_i}$$

$$\Delta T = \|\vec{u}\times\vec{v}\| = \sqrt{f_x^2 + f_y^2 + 1^2}\ \Delta A_i$$

$$\text{Surface} \approx \Sigma \Delta S_i \approx \Sigma \sqrt{f_x(x_i,y_i)^2 + f_y(x_i,y_i)^2 + 1}\ \Delta A_i$$

$$\longrightarrow S = \iint_R \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\ dA$$

Ex Surface area of the portion of the plane $z = 2 - x - y$ lying above the circle $x^2+y^2 \le 1$ in the first quadrant

Sol $S = \iint_R \sqrt{(-1)^2 + (-1)^2 + 1}\ dA$ $R = \{(x,y) \mid x^2+y^2 \le 1\}$

$= \sqrt{3} \iint_R dA = \sqrt{3} \cdot \frac{\pi}{6}$

$\underline{\underline{Ex}}$ Surface area of the paraboloid $z = 1 + x^2 + y^2$ lying above the unit circle.

Sol: $f_x = 2x, \quad f_y = 2y. \quad R = \{(x,y) \mid x^2+y^2 \le 1\}$

$$S = \iint_R \sqrt{4x^2+4y^2+1}\, dA$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\; r\,dr\,d\theta = \int_0^{2\pi} \frac{5\sqrt{5}-1}{12}\, d\theta = \frac{5\sqrt{5}-1}{12} \cdot 2\pi.$$

$\underline{\underline{Ex}}$ Surface area of the portion of the hemisphere

$$f(x,y) = \sqrt{25-x^2-y^2}$$

lying above $R = \{(x,y) \mid x^2+y^2 \le 9\}$

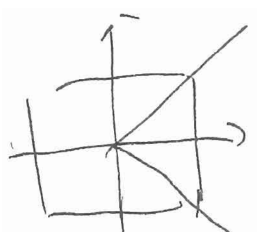
Sol: $f_x = \dfrac{-x}{\sqrt{25-x^2-y^2}}, \quad f_y = \dfrac{-y}{\sqrt{\cancel{25}-x^2-y^2}}.$

$$S = \iint_R \left(\frac{x^2}{25-x^2-y^2} + \frac{y^2}{25-x^2-y^2} + 1 \right)^{\frac{1}{2}} dA$$

$$= \int_0^{2\pi}\int_0^3 \frac{5}{\sqrt{25-r^2}}\, r\,dr\,d\theta$$

$$= 5\int_0^{2\pi} -\sqrt{25-r^2}\,\Big|_0^3\, d\theta = 10\pi.$$

$\underline{\underline{Ex}}$ $f(x,y) = 2 - x^2 - y^2 \qquad R = \{(x,y) : -1 \le x \le 1,\ -1 \le y \le 1\}$



$$R = \{(r,\theta) \mid 0 \le r \le \sec\theta,\ -\tfrac{\pi}{4} \le \theta \le \tfrac{\pi}{4}\}$$

$$\frac{1}{4} S = \iint_R \sqrt{(-2x)^2 + (-2y)^2 + 1} \; dA$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\sec\theta} \sqrt{1+4r^2} \; r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{12} (1+4r^2)^{\frac{3}{2}} \Big|_0^{\sec\theta} \; d\theta$$

not elementary → stop

Def $w = f(x, y, z)$

Q = a bound solid region

$$\sum_i f(x_i, y_i, z_i) \Delta V_i \longrightarrow \iiint_Q f(x, y, z)\, dV$$

Thm Fubini theorem If $Q = \{(x, y, z) ;\ a \le x \le b,\ h_1(x) \le y \le h_2(x)$

$g_1(x, y) \le z \le g_2(x, y)\}$

$$\iiint_Q f(x, y, z)\, dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

Ex Evaluate the triple iterated integral

$$\int_0^2 \int_0^x \int_0^{x+y} e^x (y + 2z)\, z\, dy\, dx$$

$$= \int_0^2 \int_0^x e^x (x^2 + 3xy + 2y^2)\, dy\, dx$$

$$= \frac{19}{6} \int_0^2 x^3 e^x\, dx$$

$$= \frac{19}{6} e^x (x^3 - 3x^2 + 6x - 6) \Big|_0^2$$

If $f(x,y,z)$ dV

$$\iiint_Q f(x,y,z)\,dV = \iiint_Q dV = \text{volume of } Q.$$

Ex volume of the ellipsoid given by $4x^2+4y^2+z^2=16$

$$Q=\{(x,y,z) \mid 0\le x\le 2,\ 0\le y\le \sqrt{4-x^2},\ 0\le z\le 2\sqrt{4-x^2-y^2}\}$$

$$V=\iiint_Q dV$$

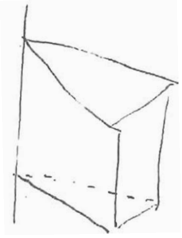
$$= 8\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{2\sqrt{4-x^2-y^2}} dz\,dy\,dx$$

$$= 16\int_0^2\int_0^{\sqrt{4-x^2}} \sqrt{4-x^2-y^2}\,dy\,dx$$

$$= 16\int_0^2 y\sqrt{4-x^2-y^2} + (4-x^2)\arcsin\left(\frac{y}{\sqrt{4-x^2}}\right)\Bigg|_0^{\sqrt{4-x^2}} dx$$

$$= 16\int_0^2 (4-x)^2 \cdot \frac{\pi}{2}\,dx$$

Ex (changing of the order of integration)



$$\int_0^{\sqrt{\frac{\pi}{2}}}\int_x^{\sqrt{\frac{\pi}{2}}}\int_1^3 \sin(y^2)\,dz\,dy\,dx$$

$$Q=\left\{(x,y,z) \,\middle|\, 0\le x\le\sqrt{\tfrac{\pi}{2}},\ x\le y\le\sqrt{\tfrac{\pi}{2}},\ 1\le z\le 3\right\}$$

$$=\left\{(x,y,z) \,\middle|\, 0\le y\le\sqrt{\tfrac{\pi}{2}},\ 0\le x\le y,\ 1\le z\le 3\right\}$$

14.7 Cylindrical & spherical

$$\iiint_Q f(x,y,z)\,dV = \iint_R \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dA$$

$$= \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} \int_{h_1(r,\theta)}^{h_2(r,\theta)} f(r,\theta,z)\,r\,dz\,dr\,d\theta$$

Ex Find the volume of the solid region Q cut from the sphere $x^2+y^2+z^2=4$ by the cylinder $r=2\sin\theta$

Sol : $V = \int_0^{\pi} \int_0^{2\sin\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$

$$= 2\int_0^{\pi} \int_0^{2\sin\theta} 2r\sqrt{4-r^2}\,dr\,d\theta$$

$$= 2\int_0^{\pi} -\frac{2}{3}(4-r^2)^{\frac{3}{2}} \Big|_0^{2\sin\theta} d\theta.$$

$$= \frac{4}{3}\int_0^{\pi} (8 - 8\cos^3\theta)\,d\theta.$$

$$\cos\theta(1-\sin^2\theta).$$

Ex $f(x,y,z) = kz$, $Q = \{(r,\theta,z) \mid 0 \le \theta \le 2\pi,\ 0 \le r < 2,\ 0 \le z \le \sqrt{16-4x^2-4y^2}\}$

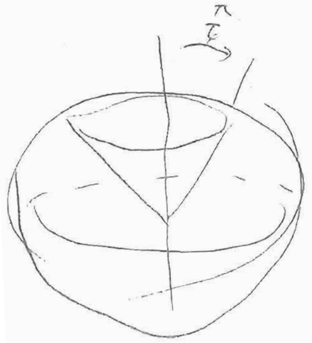
$$\int_0^{2\pi} \int_0^2 \int_0^{\sqrt{16-4r^2}} kzr\,dz\,dr\,d\theta = 16\pi k$$

$$\Delta V = (\Delta \rho_i)(\rho_i \Delta \theta_i)(\rho_i \sin \phi_i \Delta \theta_i)$$

$$\iiint_Q f(x,y,z)\,dV = \iiint_Q f(\rho,\theta,\phi)\ \rho^2 \sin\phi \,d\rho\, d\phi\, d\theta$$

$$= \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2}$$

<u>Ex</u> Find the volume of the solid region Q bounded below by the upper nappe of the cone $z^2 = x^2+y^2$ and above by the sphere $x^2+y^2+z^2 = 9$.



$x^2+y^2+z^2 = 9 \quad \Rightarrow \rho = 3$

the intersection of $z^2 = x^2+y^2$ and $x^2+y^2+z^2 = 9$

is $\quad z^2+z^2 = 2z^2 = 9$

i.e. $\quad z = \frac{3}{\sqrt{2}}$

$$z = \rho\cos\phi \Rightarrow \frac{3}{\sqrt{2}} = 3\cos\phi \Rightarrow \cos\phi = \frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{\pi}{4}$$

$$V = \iiint_Q dV = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^3 \rho^2 \sin\phi \,d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} 9\sin\theta \,d\phi\, d\theta$$

$$= 9\int_0^{2\pi} \left(1 - \frac{\sqrt{2}}{2}\right) d\theta$$

14 ~ 7 ~ 2

14.8

$(x, y) \longrightarrow (u, v)$

$x = x(u, v)$

$y = y(u, v)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Jacobian of x and y with respect to u, v

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

absolute value

pf.

$$\vec{MN} = \frac{\partial x}{\partial u}\Delta u\,\vec{i} + \frac{\partial y}{\partial u}\Delta u\,\vec{j}$$

$$\vec{MQ} = \frac{\partial x}{\partial v}\Delta v\,\vec{i} + \frac{\partial y}{\partial v}\Delta v\,\vec{j}$$

$$\vec{MN} \times \vec{MQ} = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u\,\Delta v$$

absolute value.

$$\Delta A \approx \| \vec{MN} \times \vec{MQ} \| \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u\,\Delta v$$

Ex

R = region bounded by $x-2y=0$, $x-2y=-4$, $x+y=1$, $x+y=4$,

$= \{(x,y) \mid -4 \le x-2y \le 0,\ 1 \le x+y \le 4\}$

$x - 2y = u$, $x + y = v$ $\Rightarrow$ $3x = u + 2v$, $3y = -u + v$ $\Rightarrow$ $x = \frac{1}{3}(u + 2v)$, $y = \frac{1}{3}(-u + v)$

$\frac{\partial x}{\partial u} = \frac{1}{3}$, $\frac{\partial x}{\partial v} = \frac{2}{3}$, $\frac{\partial y}{\partial u} = -\frac{1}{3}$, $\frac{\partial y}{\partial v} = \frac{1}{3}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

$$\iint_R 3xy \, dA = \int_{-4}^{0} \int_{1}^{4} 3 \cdot \frac{1}{3}(u+2v)\,(+\tfrac{1}{3})(-u+v)\, \left|\tfrac{1}{3}\right| \, dv\, du$$

$$= (+\tfrac{1}{9}) \int_{-4}^{0} \int_{1}^{4} (-u^2 - uv + 2v^2)\, dv\, du$$

$$= (+\tfrac{1}{9}) \int_{-4}^{0} \left(-u^2 v - u\frac{v^2}{2} + \frac{2}{3}v^3\right)\Big|_1^4 du$$

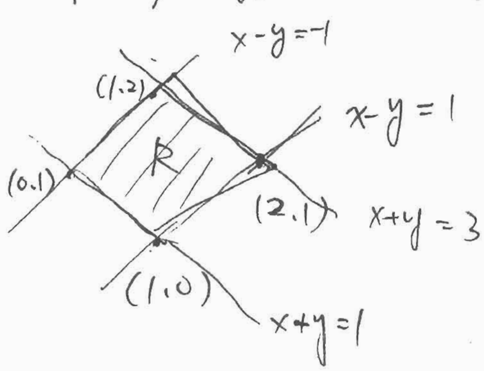
$$= +\frac{1}{9}\int_{-4}^{0} \left(-3u^2 - u\frac{15}{2} + \frac{2}{3}\cdot 63\right) du$$

$-\frac{15}{4}\cdot 16$

$$= +\frac{1}{9}\left(-u^3 - \frac{15}{4}u^2 + 42u\right)\Big|_{-4}^{0}$$

$$= \frac{+1}{9}(-64 + 60 + 168) = \frac{164}{9}$$

Ex $R = \{(x,y) : 1 \le x+y \le 3, \; -1 \le x-y \le 1\}$



$x+y = v$
$x-y = u$
$\Rightarrow$ $2x = u+v$
$2y = -u+v$

$\Rightarrow x = \frac{1}{2}(u+v)$

$y = \frac{1}{2}(-u+v)$

$14 - \cancel{16}\,8 - 2$

$S = \{(u, v) \mid -1 \le u \le 1 ,\ 1 \le v \le 3\}$

$$\iint_R (x+y)^2 \sin^2(x-y)\, dA$$

$$\frac{\partial x}{\partial u} = \frac{1}{2} \quad \frac{\partial x}{\partial v} = \frac{1}{2} ,\quad \frac{\partial y}{\partial u} = -\frac{1}{2} ,\quad \frac{\partial y}{\partial v} = \frac{1}{2}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$= \int_{-1}^{1} \int_{1}^{3} v^2 \sin^2 u \left|\frac{1}{2}\right| dv\, du$$

$$= \frac{1}{2} \int_{-1}^{1} \frac{v^3}{3} \sin^2 u \Big|_1^3 du$$

$$= \frac{1}{2} \int_{-1}^{1} \frac{26}{3} \cdot \frac{1 - \cos 2u}{2}\, du$$

$$= \frac{13}{6} \left(u - \frac{1}{2} \sin 2u \Big|_{-1}^{1} \right)$$

$$= \frac{13}{6} \left[\left(1 - \frac{1}{2} \sin 2\right) - \left(-1 - \frac{1}{2} \sin -2\right) \right]$$

$$= \frac{13}{6} (2 - \sin 2)$$