① $z = f(x,y)$

x, y independent variable

z dependent variable.

② ~~level curves for~~

Scalar field = $(x,y) \longmapsto z = f(x,y)$

level curves = $f(x,y) = \text{const}$

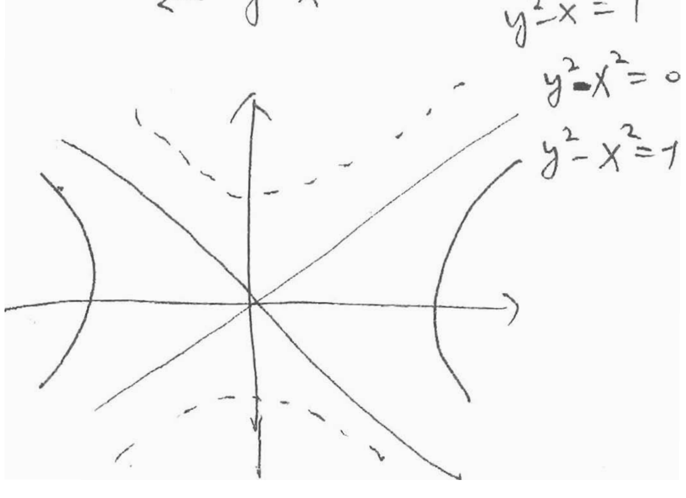
or contour lines

— isobars
— isotherm
— equipotential

$(x,y,z) \longmapsto w = f(x,y,z)$

level surface = $f(x,y,z) = \text{const}$.

$z = y^2 - x^2$



$f(x,y,z) = 4x^2 + y^2 + z^2$

Def $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$

$\Leftrightarrow \forall \varepsilon>0 \ \exists \delta>0, \ |f(x,y)-L| < \delta \quad \forall \ |(x,y)-(x_0,y_0)| < \varepsilon.$

Ex $\lim_{(x,y)\to(1,2)} \frac{5x^2y}{x^2+y^2}$

Ex $\lim_{(x,y)\to(0,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$ does not exist !

$y=0 \quad \lim_{(x,0)\to(0,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2 = \lim_{x\to0} \frac{x^2}{x^2} = 1$

$x=y \quad \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)^2}{(x^2+y^2)} = \lim_{x\to0} \left(\frac{x^2-x^2}{x^2+x^2}\right)^2 = 0$

Def $f(x,y)$ conti at (x_0,y_0)

$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0),$

or

$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0)$

13-3

Def

$$\frac{\partial f(x,y)}{\partial x} = f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\frac{\partial f(x,y)}{\partial y} = f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Ex $f(x,y) = xe^{x^2 y}$. ~~at (1, 2)~~

$f_x(x,y) = e^{x^2 y} + x \cdot e^{x^2 y}(2xy)$

$f_y(x,y) = xe^{x^2 y} \cdot x^2$

at $(x,y) = (1, \ln 2)$

Ex $f(x,y) = -\frac{x^2}{2} - y^2 + \frac{25}{8}$

Def $f(x_1, x_2, \cdots x_n)$

$$f_{x_i} = \frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1 \cdots, x_i + \Delta x_i, \cdots x_n) - f(x_1 \cdots x_n)}{\Delta x_i}$$

Def

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial y} = f_{yy}$$

Ex $f(x,y) = 3xy^2 - 7y + 5x^2y^2$

$f_{xy}(-1,2) = ?$

Problem : when $f_{xy} = f_{yx}$

Thm : if f_{xy}, f_{yx} continuous at (x_0, y_0)

$\Rightarrow f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$

Moreover if f_{xy}, f_{yx} continuous $\forall (x,y) \in R$,

$\Rightarrow f_{xy}(x,y) = f_{yx}(x,y) \quad \forall (x,y) \in R$

Ex $f(x,y,z) = ye^x + x\ln z$

$\Rightarrow f_{xzz} = f_{z\cancel{x}z} = f_{zzx}$

Def $z = f(x,y)$

$$\begin{cases} dx = \Delta x \\ dy = \Delta y \end{cases}$$ differential of independent variable

$$\begin{cases} \Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) \\ \Delta x \\ \Delta y \end{cases}$$ increment of x, y, z.

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$ total differential of the dependent variable z

Ex ① $z = 2x\sin y + 3x^2y^2$.

$\Rightarrow dz$

② $w = x^2+y^2+z^2$

$\Rightarrow dw$

Def $f(x,y)$ differentiable at (x_0, y_0)

if $\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$.

with $\displaystyle\lim_{(\Delta x,\Delta y)\to 0} \varepsilon_1 = 0 = \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_2$

Thm If f_x, f_y continuous on ~~R~~ D

$\Rightarrow f$ differentiable on ~~R~~ D

Ex

① $f(x,y) = x^2+3y$

$f(x+\delta x,\ y+\delta y) - f(x,y)$

$= 2x\cdot\delta x + 3(\delta y) + \delta x\cdot(\delta x) + 0\cdot(\delta y)$

$f_x = 2x$

$f_y = 3.$

② $f(x,y) = \begin{cases} \dfrac{-3xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

$f_x(0,0) = \lim_{(0+\delta x,0)\to(0,0)} \dfrac{\dfrac{-3(\delta x)\cdot 0}{(0+\delta x)^2+0^2} - 0}{\delta x} = 0$

$f_y(0,0) = \lim_{(0,0+\delta y)\to(0,0)} \dfrac{\dfrac{-3\cdot 0\cdot(0+\delta y)}{0^2+(0+\delta y)^2} - 0}{\delta y} = 0$

$\cancel{\lim f_x(x,y) = \dfrac{-3y(x^2+y^2) - \cdots}{(x^2+y^2)^2}}$

$\lim_{(x,x)\to(0,0)} = -\dfrac{3}{2}$, $\lim_{(x,-x)} f(x,y) = \dfrac{3}{2}$

$\Rightarrow$ $f(x,y)$ not conti

~~$\Rightarrow$ $f(x,y)$ not diff~~

$\cancel{dz} =$

$\Delta z \approx dz$ linear approximation

Ex $z = \sqrt{4 - x^2 - y^2}$

$x, y : (1,1) \longrightarrow (1.01, 0.97)$

$dx = \Delta x = 1.01 - 1 = 0.01$

$dy = \Delta y = 0.97 - 1 = -0.03$

$$\Delta z = \sqrt{4 - (1.01)^2 - (0.97)^2} - \sqrt{4 - 1^2 - 1^2}$$

$$\approx dz$$

$$= \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= \left. \frac{-x}{(4 - x^2 - y^2)^{\frac{1}{2}}} \right|_{(x,y)=(1,1)} \cdot \Delta x + \left. \frac{-y}{\sqrt{4 - x^2 - y^2}} \right|_{(x,y)=(1,1)} \Delta y$$

$$= \frac{-1}{(4 - 1^2 - 1^2)^{\frac{1}{2}}} \cdot 0.01 + \frac{-1}{(4 - 1^2 - 1^2)^{\frac{1}{2}}} \cdot (-0.03)$$

$$= \frac{1}{\sqrt{2}} \cdot 0.02$$

13.5

$w = f(x, y)$

$\begin{cases} x = g(t) \\ y = \cancel{g}\, h(t) \end{cases}$

$\Rightarrow$ $w = f(x, y) = f(x(t), y(t))$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Ex $w = x^2 y - y^2$, $x = \sin t$, $y = e^t$

$$\frac{dw}{dt} = 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}$$

Ex $(x_1, y_1) = (4\cos t, 2\sin t)$

$(x_2, y_2) = (2\sin 2t, 3\cos t)$

$$r = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]^{\frac{1}{2}}$$

$$\left.\frac{dr}{dt}\right|_{t=\pi} = \frac{22}{5}$$

$$= \frac{\partial r}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial r}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial r}{\partial x_3}\frac{dx_3}{dt} + \frac{\partial r}{\partial x_4}\frac{dx_4}{dt}$$

Ex $w = 2xy$ $x = r^2 + t^2$, $y = \frac{r}{t}$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = 2y \cdot 2r + 2x \cdot \frac{1}{t} = \frac{6r^2 + 2t^2}{t}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = 2y \cdot 2t + 2x \cdot \left(-\frac{1}{t^2}\right) = \frac{2st^2 - 2s^3}{t^2}$$

Ex $w = xy + yz + xz$

$x = r\cos t$

$y = r\sin t$ $\qquad t = 2\pi$

$z = t$

$\frac{\partial w}{\partial r} =$

$\frac{\partial w}{\partial t} =$

Thm : ① $f(x,y) = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}$.

② $F(x.y.z) = 0$

$\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

Ex $x^2 + y^2 = 25$

Ex $3x^2 z = x^2y^2 + 2z^3 + 3yz - 5 = 0$

Def $u = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$

$$D_u f(x,y) = \lim_{t\to 0} \frac{f(x+t\cos\theta,\ y+t\sin\theta) - f(x,y)}{t}$$

directional derivative of f in the direction of u

Thm f differentiable $\Rightarrow D_u f = \frac{\partial f}{\partial x}\cdot\cos\theta + \frac{\partial f}{\partial y}\cdot\sin\theta$

pf

$$g(t) = f(x_0+t\cos\theta,\ y_0+t\sin\theta)$$

$$\begin{aligned} D_u f(x,y) &= g'(t)\big|_0 \\ &= \left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t}\right)\Big|_{t=0} + \left(\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}\right)\Big|_{t=0} \\ &= f_x(x_0,y_0)\cdot\cos\theta + f_y(x_0,y_0)\sin\theta \end{aligned}$$

Ex $f(x,y) = x^2\sin 2y$ ~~at~~

$V = 3i - 4j \Rightarrow u = \frac{3}{5}i - \frac{4}{5}j$

at $(1, \frac{\pi}{2})$

$$D_u f(1,\tfrac{\pi}{2}) = (2x\sin 2y)\cos\theta + (2x^2\cos 2y)(\sin\theta) = \frac{8}{5}$$

Def $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ gradient of f

$$= \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

Ex $f(x,y) = y \ln x + xy^2$

$\nabla f = \left(\frac{y}{x} + y^2\right) \vec{i} + (\ln x + 2xy) \vec{j}$

Thm f differentiable

$\Rightarrow D_u f = \vec{\nabla} f \cdot \vec{u}$

Thm f differentiable

$\Rightarrow$ (i) $\nabla f(x,y) = 0 \Rightarrow D_u f(x,y) = 0$

(ii) The direction of maximum increase of f is given by $\vec{\nabla} f$.

(iii) The direction of minimum increase of f is given by $-\vec{\nabla} f$.

Ex $T(x,y) = 20 - 4x^2 - y^2$

$r'(t) = \vec{\nabla} T$

$r(0) = (2, -3)$

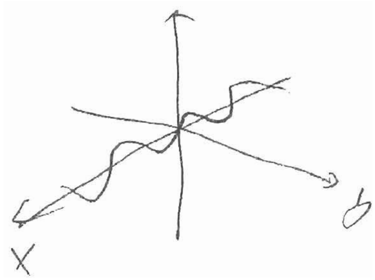
$\Rightarrow x = \frac{2}{81} y^4$

Thm $f(x, y)$ differentiable

$\nabla f(x_0, y_0) \neq 0$

$\Rightarrow \nabla f(x_0, y_0) \perp$ level curve through (x_0, y_0)

Ex $f(x, y) = y - \sin x$



$\Leftarrow$ ~~level~~ $y - \sin x = 0$ level curve

$\nabla f = -\cos x \ \vec{i} + 1 \ \vec{j}$

Def $w = f(x, y, z)$

$\vec{\nabla} f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$

$\vec{u} = a \vec{i} + b \vec{j} + c \vec{k}$, $\|\vec{u}\| = 1$

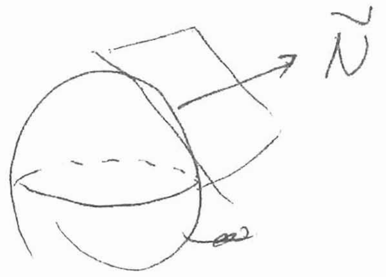
$\Rightarrow$ ① $D_u f = \vec{\nabla} f \cdot \vec{u}$

② $\nabla f = 0 \Rightarrow D_u f = 0$

Ex $f(x, y, z) = x^2 + y^2 - 4z$ at $(2, -1, 1)$

$F(x^2, y, z) = x^2 + y^2 + z^2 - x$

$F(x, y, z) = 0$



$r(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$ on surface.

$F(x(t), y(t), z(t)) = 0$

$\Rightarrow 0 = \dfrac{\partial F}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial F}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial F}{\partial z}\dfrac{dz}{dt} = 0$

$= \vec{\nabla} F \cdot r'(t)$

Def ∇F normal vector to the tangent plane

Thm $F(x, y, z)$ differentiable at (x_0, y_0, z_0)

$\Rightarrow F_x(x_0, y_0, z_0)\,(x - x_0) + F_y(x_0, y_0, z_0)\cdot(y - y_0)$

$+ F_z(x_0, y_0, z_0)\,(z - z_0) = 0$

eq of ~~tangent~~ the tangent plane.

Ex $z^2 - 2x^2 - 2y^2 = 12$ at $(1, -1, 4)$

Ex tangent line to the curve. of the intersection of the surface

$$\begin{cases} x^2+2y^2+2z^2=20 \\ x^2+y^2+z=4 \end{cases}$$

at $(0,1,3)$

$$\vec{\nabla} F \times \vec{\nabla} G = \begin{vmatrix} i & j & k \\ 0 & 4 & 12 \\ 0 & 2 & 1 \end{vmatrix} = -20\, i$$

Ex angle of inclination of the tangent plane of

$$\frac{x^2}{12}+\frac{y^2}{12}+\frac{z^2}{3}=1$$

at $(2,2,1)$

13.8

Thm : R bounded, closed

$f(x,y)$ conti on R

$\Rightarrow f(x,y)$ has maximum & minimum on R

Def f conti on R, $(x_0, y_0) \in R$

① f has a relative minimum at (x_0, y_0) if

$$f(x,y) \geq f(x_0, y_0)$$

for all (x,y) in a ~~B~~ open ball centered at (x_0, y_0)

② f has a relative maximum at (x_0, y_0) if

$$f(x,y) \leq f(x_0, y_0)$$

for all (x,y) " "

relative extrema = relative minimum or relative maximum

Def (x_0, y_0) critical point

if ① $\frac{\partial f}{\partial x}(x_0, y_0) = 0 = \frac{\partial f}{\partial y}(x_0, y_0)$

or

② $\frac{\partial f}{\partial x}(x_0, y_0)$ or $\frac{\partial f}{\partial y}(x_0, y_0)$ does not exist.

Thm f relative extrema at (x_0, y_0)

$\Rightarrow (x_0, y_0)$ critical point of f

Ex $\quad f(x,y) = 2x^2 + y^2 + 8x - 6y + 20$ $\quad$ rel extrema

$$\begin{cases} \dfrac{\partial f}{\partial x} = 4x + 8 = 0 \\ \dfrac{\partial f}{\partial y} = 2y - 6 = 0 \end{cases} \Rightarrow \begin{cases} x = -2 \\ y = 3 \end{cases} \quad \text{critical point}$$

$$f(x,y) = 2(x+2)^2 + 3(y-3)^2 + 3 > 3 = f(2,3)$$

$\Rightarrow$ $f(2,3) = 3$ $\quad$ rel min

Ex $\quad f(x,y) = 1 - (x^2+y^2)^{\frac{1}{3}}$ $\quad$ rel extrema

$$\frac{\partial f}{\partial x} = \frac{-2x}{3\,(x^2+y^2)^{\frac{2}{3}}} = 0$$

$$\frac{\partial f}{\partial y} = \frac{-2y}{3\,(x^2+y^2)^{\frac{2}{3}}} = 0$$

$\Rightarrow$ $(x,y) = (0,0)$ the only critial point

$$f(x,y) = 1 - (x^2+y^2)^{\frac{1}{3}} < 1$$

$\Rightarrow$ $f(0,0) = 1$ $\quad$ rel max

Thm $\quad$ f has conti second derivatives.

$f_x(a,b) = 0 = f_y(a,b)$

$$d = f_{xx} f_{yy} - f_{xy}^2$$

$\Rightarrow$ (i) $d > 0$ $\quad f_{xx}(a,b) > 0$ $\quad$ rel min (a,b)

(ii) $d > 0$ $\quad f_{xx}(a,b) < 0$ $\quad$ rel max

(iii) $d < 0$ $\quad$ saddle point $(a,b,f(a,b))$

(iv) $d = 0$, inclusive

13-8-2

$$f_{xx}\cdot(\Delta x)^2+2f_{xy}(\Delta x)(\Delta y)+f_{yy}(\Delta y)^2$$

$$=f_{xx}\left[(\Delta x)^2+2\frac{f_{xy}}{f_{xx}}(\Delta x)(\Delta y)+\left(\frac{f_{xy}}{f_{xx}}\right)^2(\Delta y)^2\right]+\left[f_{yy}-\frac{(f_{xy})^2}{f_{xx}}\right](\Delta y)^2$$

$$=f_{xx}\left[\Delta x+\frac{f_{xy}}{f_{xx}}\Delta y\right]^2+\left(\frac{f_{xx}f_{yy}-f_{xy}^2}{f_{xx}}\right)(\Delta y)^2$$

$$=f_{xx}\left[\left(\Delta x+\frac{f_{xy}}{f_{xx}}\Delta y\right)^2+\frac{f_{xx}f_{yy}-f_{xy}^2}{(f_{xx})^2}(\Delta y)^2\right]$$

Ex $f(x,y)=-x^3+4xy-2y^2+1$

$f_x(x,y)=-3x^2+4y$ $\Rightarrow (x,y)=(0,0)$ or $\left(\frac{4}{3},\frac{4}{3}\right)$

$f_y(x,y)=4x-4y$ $\Rightarrow x=y$.

$f_{xx}=-6x$, $f_{xy}=4$, $f_{yy}=-4$

$d(x,y)=-6x\cdot(-4)-4^2$

$d(0,0)=-16<0$

$\rightarrow (0,0,1)$ saddle point

$d\left(\frac{4}{3},\frac{4}{3}\right)=-24\cdot\frac{4}{3}-16=16>0$

$f_{xx}\left(\frac{4}{3},\frac{4}{3}\right)=-6\cdot\frac{4}{3}<0$

$\Rightarrow$ rel maxi at $\left(\frac{4}{3},\frac{4}{3}\right)$

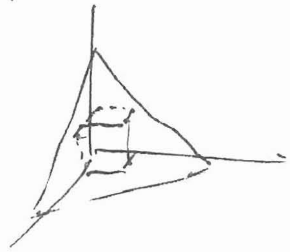
Ex $f(x,y)=x^2y^2$ $\Rightarrow$ $x=0$ or $y=0$

$d(x,y)=0$

$\Rightarrow$ Fail, But $f(x,y)\geq 0$ rel min ~~[illegible]~~

13-9

<u>Ex</u> $6x+4y+3z=24$



$V(x,y) = x \cdot y \cdot \left(\frac{1}{3}(24-6x+4y)\right) = \frac{1}{3}xy(24-6x+4y)$

$\frac{\partial V}{\partial x} = \frac{1}{3}y(24-6x+4y) + \frac{1}{3}xy \cdot (-6)$

$= \frac{1}{3}y(24-12x-4y) = 0 \quad \Rightarrow \quad y=0 \text{ or } 24-12x-4y=0$

$\frac{\partial V}{\partial y} = \frac{1}{3}x(24-6x+4y) + \frac{1}{3}xy \cdot 4$

$= \frac{1}{3}x(24-6x+8y) = 0 \quad \Rightarrow \quad x=0 \text{ or } 24-6x+8y=0$

critical points: $(0,0)$, $(0,6)$, $(4,0)$, $(\frac{4}{3}, 2)$

At points $(0,0)$, $(0,6)$, $(4,0)$ $V(x,y)=0 \Rightarrow$ not max.

$V_{xx} = -4y$, $V_{xy} = \frac{1}{3}(24-12x-4y) + \frac{1}{3}y(-4) = \frac{1}{3}(24-12x-8y)$

$V_{yy} = -\frac{8}{3}x$

$d\Big|_{(\frac{4}{3},2)} = (-4)\cdot(2)\cdot(-\frac{8}{3})\cdot(\frac{4}{3}) - \left[\frac{1}{3}(-24-12\cdot\frac{8}{3}-8\cdot 2)\right]^2$

$= \frac{64}{3} > 0$

$V_{xx} = -4 \cdot 2 < 0$

$\Rightarrow V(\frac{4}{3}, 2)$ max

Thm B. least square regression line for

$(x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)$

is given by $f(x) = ax+b$

where

$$a = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2}, \quad b = \frac{1}{n}\left(\sum y_i - a \sum_{i=1}^{n} x_i\right)$$

pf

$$S(a,b) = \sum_{i=1}^{n} [f(x_i) - y_i]^2$$

$$= \sum_{i=1}^{n} (ax_i + b - y_i)^2$$

$$\frac{\partial S}{\partial a} = \sum_{i=1}^{n} 2(ax_i + b - y_i) \cdot x_i = 0$$

$$\frac{\partial S}{\partial b} = \sum_{i=1}^{n} 2(ax_i + b - y_i) = 0$$

$$\Rightarrow \left(\cancel{2}\sum_{i=1}^{n} x_i^2\right) a + \left(\cancel{2}\sum_{i=1}^{n} x_i\right) b = \sum_{i=1}^{n} \cancel{2}\, x_i y_i$$

$$\left(\cancel{2} \sum_{i=1}^{n} x_i\right) a + \cancel{2} n b = \cancel{2} \sum_{i=1}^{n} y_i$$

$$a = \frac{\cancel{4} n \sum x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum y_i\right)}{\cancel{4} n \cdot 2 \left(\sum_{i=1}^{n} x_i\right)^2 - \cancel{4} \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i\right)}$$

$$b = \frac{1}{n}\left[\sum_{i=1}^{n} y_i - a \cdot \left(\sum_{i=1}^{n} x_i\right)\right]$$

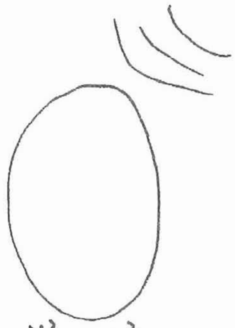
ex40

$$\frac{\partial^2 S}{\partial a^2} = 2 \sum_{i=1}^{n} x_i^2, \quad \frac{\partial^2 S}{\partial a \partial b} = 2 \sum x_i, \quad \frac{\partial^2 S}{\partial b \partial b} = 2n.$$

$$\Rightarrow d = 4n\left(\sum_{i=1}^{n} x_i^2\right) - 4\left(\sum x_i\right)^2 < 0 \quad \not\Rightarrow \text{ min.}$$

$$\frac{\partial^2 S}{\partial a^2} > 0$$

Ex $f(x,y) = 4xy$ under $\frac{x^2}{3} + \frac{y^2}{4} = 1 = g(x,y)$



~~thm~~ $\Rightarrow \vec{\nabla} f = \lambda \nabla g$

thm $f \cdot g$ with conti first derivatives

f has extreme at (x_0, y_0) on the smooth contraint curve $g(x,y) = c$.

If $\nabla g(x_0, y_0) \neq 0$, then

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

for some λ

pf $g(x,y) = c$ represented by $r(t) = x(t)\vec{i} + y(t)\vec{j}$, $r'(t) \neq 0$.

$h(t) = f(x(t), y(t))$ has extreme at (x_0, y_0)

$\Rightarrow \left.\frac{dh(t)}{dt}\right|_{(x_0,y_0)} = 0$ i.e. $f_x \cdot x' + f_y \cdot y' \Big|_{(x_0,y_0)} = 0$

$\Rightarrow \vec{\nabla} f(x_0, y_0) \cdot (x'(x_0, y_0), y'(x_0, y_0)) = 0 \Rightarrow \vec{\nabla} f \perp \vec{r}'(x_0, y_0)$

$\Rightarrow$ ~~$\vec{\nabla} f = \lambda \vec{\nabla} g$~~ But $\vec{\nabla} g \perp \vec{r}'(x_0, y_0)$

$\Rightarrow \vec{\nabla} f \parallel \vec{\nabla} g \Rightarrow \vec{\nabla} f = \lambda \nabla g$

Ex 1 Maxi $f(x,y) = 4xy$ under $x>0$, $y>0$, $g(x) = \frac{x^2}{3^2} + \frac{y^2}{4^2} = 1$.

$$f_x = 4y = \lambda g_x = \lambda \cdot \frac{2x}{9} \quad (1)$$

$$f_y = 4x = \lambda g_y = \lambda \cdot \frac{2y}{16} \quad (2)$$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1 \quad (3)$$

$$\frac{(1)}{(2)} \Rightarrow \frac{y}{x} = \frac{\frac{2x}{9}}{\frac{2y}{16}} = \frac{16}{9}\frac{x}{y} \Rightarrow y^2 = \frac{16}{9}x^2$$

$$\frac{x^2}{9} + \frac{1}{16} \cdot \frac{16}{9}x^2 = 1 \Rightarrow 2x^2 = 9 \Rightarrow x = \frac{3}{\sqrt{2}}$$

$$y^2 = \frac{16}{9} \cdot \frac{9}{2} = 8 \Rightarrow y = 2\sqrt{2}$$

$$f(x,y) = 4 \cdot \frac{3}{\sqrt{2}} \cdot 2\sqrt{2} = 24$$

Ex2 Minimum ~~f(x,y,z) = 2x~~ $f(x,y,z) = 2x^2 + y^2 + 3z^2$

$$g(x,y,z) = 2x - 3y - 4z = 49$$

$\Rightarrow$ at $(3, -9, -4)$

Ex 3 Extreme $f(x,y) = x^2 + 2y^2 - 2x + 3$ under $x^2 + y^2 \leq 10$

Sol : (a) $g(x,y) = x^2 + y^2 - 10$

$$f_x = 2x - 2 = \lambda \cdot g_x = \lambda \cdot (2x) \quad (1)$$

$$f_y = 4y = \lambda g_y = \lambda \cdot 2y \quad (2)$$

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$x^2+y^2=10 \qquad (3)$

(2) $\Rightarrow \lambda = 2$ or $y=0$ $\qquad \lambda = ? \Rightarrow x = \sqrt{10}$

(1) $\Rightarrow$ $\cancel{2x-2 = 2(2x)} \Rightarrow \cancel{2x=4}$ $\Rightarrow$

$2x-2 = 2-2x \Rightarrow x=-1$

(3) $y^2=9 \Rightarrow y = \pm 3.$

$f(-1,3) = (-1)^2+2\cdot 3^2 - 2\cdot(-1)+3 = 24 = f(-1,-3)$

(b) $x^2+y^2 < 10$

$f_x = 2x-2 = 0$
$f_y = 4y = 0$ $\Rightarrow$ $x=1$, $y=0$ $\Rightarrow$ $1^2+0^2 < 10$

$f(1,0) = 1^2+2\cdot 0^2 - 2 + 3 = 2$

$f_{xx} = 2.\ f_{yy}=4 \quad f_{xy} = 0 \qquad d = 2\cdot 4 - 0^2 = 8 > 0$

$f(1,0)$ mini

Conclusion f has a maxi 24 at $(-1,\pm 3)$
mini 2 at $(1,0)$

Ex $T(x,y,z)$ Two constraint : $\nabla f = \lambda \nabla g + \mu \nabla h$

Ex $T(x,y,z) = 2x+2y+z^2+20$

$g(x,y,z) = x^2+y^2+z^2 = 11 \qquad (4)$

$h(x,y,z) = x+y+z=3 \qquad (5)$

$T_x = 2 = \lambda g_x + \mu h_x = \lambda\cdot 2x + \mu\cdot 1 \qquad (1)$

$T_y = 2 = \lambda g_y + \mu h_y = \lambda\cdot 2y + \mu\cdot 1 \qquad (2)$

$T_z = 2z = \lambda g_z + \mu h_z = \lambda\cdot 2z + \mu\cdot 1 \qquad (3)$

$(1)-(2) \Rightarrow \lambda(x-y)=0$

① $\lambda=0$

$(2) \Rightarrow \mu=2$

$(3) \Rightarrow z=1$

$(5) \Rightarrow y=2-x$

$(1) \quad x^2+(2-x)^2+1^2=11 \Rightarrow 2x^2-4x-6=0 \Rightarrow x^2-2x-3=0$

$\Rightarrow x=-1, 3$

critial point $(-1, 3, 1)$, $(3, -1, 1)$

② $x=y$

$\cancel{2x^2+z^2}=11$

$(5) \Rightarrow 2x+z=3 \Rightarrow z=3-2x$

$(4) \Rightarrow x^2+y^2+(3-2x)^2=11 \Rightarrow 6x^2-12x-2=0$

$\Rightarrow 3x^2-6x-1=0$

$\Rightarrow x=\frac{3\pm\sqrt{12}}{3}=\frac{3\pm2\sqrt{3}}{3}$

$\Rightarrow z=\frac{3\mp4\sqrt{3}}{3}$

critial point $\left(\frac{3+2\sqrt{3}}{3}, \frac{3+2\sqrt{3}}{3}, \frac{3-4\sqrt{3}}{3}\right)$

$\left(\frac{3-2\sqrt{3}}{3}, \frac{3-2\sqrt{3}}{3}, \frac{3+4\sqrt{3}}{3}\right)$

$T(-1,3,1)=25=T(-1,3,1)$ min

$T\left(\frac{3+2\sqrt{3}}{3}, \frac{3+2\sqrt{3}}{3}, \frac{3+4\sqrt{3}}{3}\right)=\frac{91}{3}$ max

$T\left(\frac{3-2\sqrt{3}}{3}, \frac{3-2\sqrt{3}}{3}, \frac{3+4\sqrt{3}}{3}\right)=\frac{91}{3}$