Def: vector-valued function

$$\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j}$$

or $$\vec{r}(t) = f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$$

Ex ~~r(t) = 4 cos t i~~

$$r(t) = 4\cos t\,\vec{i} + 4\sin t\,\vec{j} + t\,\vec{k}$$

Ex

$$\begin{cases} \dfrac{x^2}{12} + \dfrac{y^2}{24} + \dfrac{z^2}{4} = 1 \\ y = x^2 \end{cases}$$

$$\Rightarrow r(t) = t\,\vec{i} + t^2\,\vec{j} + \sqrt{\frac{24 - 2t^2 - t^4}{6}}\,\vec{k}$$

Def

$$\lim_{t\to a} \vec{r}(t) = \left(\lim_{t\to a} f(t)\right)\vec{i} + \left(\lim_{t\to a} g(t)\right)\vec{j} + \left(\lim_{t\to a} h(t)\right)\vec{k}$$

Def $\vec{r}(t)$ cont at $t=a$

If $\lim_{t\to a} \vec{r}(t) = \vec{r}(a)$

Ex

$$r(t) = \begin{cases} t^2\,\vec{i} + 3t\,\vec{j} + \dfrac{\sin t}{t}\,\vec{k}, & t=0 \\ \vec{0} & t=0 \end{cases}$$

12-2

$$\vec{r}'(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}\,\vec{i} + \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}\,\vec{j} + \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t}\,\vec{k}$$

$$= f'(t)\,\vec{i} + g'(t)\,\vec{j} + h'(t)\,\vec{k}$$

Ex $\vec{r}(t) = \sin t\,\vec{i} + e^{t^2}\,\vec{j} + \ln(t^3-1)\,\vec{k}$

$$\vec{r}'(t) = \cos t\,\vec{i} + 2t e^{t^2}\,\vec{j} + \frac{3t^2}{t^3-1}\,\vec{k}$$

$$\vec{r}''(t) = -\sin t\,\vec{i} + (4t^2 e^{t^2} + 2e^{t^2})\,\vec{j} + \frac{6t(t^3-1) - 9t^4}{(t^3-1)^2}\,\vec{k}$$

$$\vec{r}'(t) \cdot \vec{r}''(t) = \cancel{\sin t}(+\cos t)(-\sin t) + 2t e^{t^2}(4t^2 e^{t^2} + 2e^{t^2}) + \frac{3t^2(-3t^4 - 6t)}{(t^3-1)^3}$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} i & j & k \\ \cos t & 2t e^{t^2} & \frac{3t^2}{t^3-1} \\ -\sin t & (4t^2+2)e^{t^2} & \frac{-3t^4-6t}{(t^3-1)^2} \end{vmatrix}.$$

$$=$$

Thm $\frac{d}{dt}\left[f(t)\cdot\vec{r}(t)\right] = f'(t)\,\vec{r}(t) + f(t)\,\vec{r}'(t)$

$\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{u}(t)\right] = \vec{r}'(t)\cdot\vec{u}(t) + \vec{r}(t)\cdot\vec{u}'(t)$

$\frac{d}{dt}\left[\vec{r}(t)\times\vec{u}(t)\right] = \vec{r}'(t)\times\vec{u}(t) + \vec{r}(t)\times\vec{u}'(t)$

$\vec{r}(t)\cdot\vec{r}(t) = \text{const} \quad\Rightarrow\quad \vec{r}(t)\cdot\vec{r}'(t) = 0$

Thm

$$\int \vec{r}(t)\,dt = \int \left(f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}\right)dt$$
$$= \left(\int f(t)\,dt\right)\vec{i} + \left(\int g(t)\,dt\right)\vec{j} + \left(\int h(t)\,dt\right)\vec{k}$$

Ex

$$\int_0^1 \left(\sqrt[3]{t}\,\vec{i} + \frac{1}{t+1}\,\vec{j} + e^{-t}\,\vec{k}\right)dt$$

Ex

$\vec{r}'(t) = \cos 2t\,\vec{i} + -2\sin t\,\vec{j} + \frac{1}{1+t^2}\,\vec{k}$, $r(0) = 3\vec{i} - 2\vec{j} + \vec{k}$.

$\Rightarrow r(t) = \left(\frac{1}{2}\sin 2t + 3\right)\vec{i} + (2\cos t - 4)\,\vec{j} +$

$(\arctan t + 1)\,\vec{k}$

$(2-2^{-2}$

12.3

$$r(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

$$= (x(t),\ y(t),\ z(t))$$

$$\vec{v}(t) = r'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}'$$

$$= (x'(t),\ y'(t),\ z'(t))$$

$$\vec{a}(t) = \vec{v}'(t) = x''(t)\vec{i} + y''(t)\vec{j} + z''(t)\vec{k}''$$

$$= (x''(t),\ y''(t),\ z''(t))$$

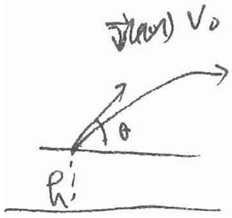
Ex

$$\vec{a}(t) = \vec{j} + 2\vec{k}$$

$$\vec{r}(0) = (1, 2, 0)$$

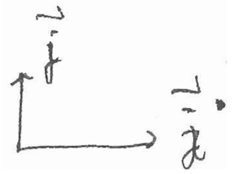
$$\Rightarrow r(t) = \vec{i} + \left(\frac{t^2}{2} + 2\right)\vec{j} + t^2\vec{k}$$

Thm



$$v(0) = v_0 \cos\theta\ \vec{i} + v_0 \sin\theta\ \vec{j}$$

$$a(t) = -\vec{g}\ \vec{k}$$



$$\Rightarrow \vec{r}(t) = (v_0\cos\theta)t\ \vec{i} + \left(h + (v_0\sin\theta)t - \frac{1}{2}gt^2\right)\vec{k}$$

12-4

Def $C: \vec{r}(t)$ smooth curve, $\|r'(t)\| \neq 0$

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$ unit tangent vector, $\|r'(t)\| \neq 0$

Ex $r(t) = t\vec{i} + t^2\vec{j}$

$r'(t) = 1\vec{i} + 2t\vec{j}$

$\|r'(t)\| = \sqrt{1^2 + (2t)^2} = \sqrt{1+4t^2}$

$$T(t) = \frac{t\vec{i} + t^2\vec{j}}{\sqrt{1+4t^2}}$$

Ex $r(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + t\vec{k}$

$$T(t) = \frac{-2\sin t\,\vec{i} + 2\cos t\,\vec{j} + \vec{k}}{\|-2\sin t\,\vec{i} + 2\cos t\,\vec{j} + \vec{k}\|}$$

Def $T'(t) \neq 0$

$$\Rightarrow N(t) = \frac{T'(t)}{\|T'(t)\|}$$ principal unit normal vector

Ex $\vec{r}(t) = 3t\,\vec{i} + 2t^2\vec{j}$

$$T(t) = \frac{3\vec{i} + 4t\vec{j}}{\sqrt{9+16t^2}} \qquad N(t) = \frac{-4t\vec{i} + 3\vec{j}}{\sqrt{9+16t^2}}$$

Thm $\vec{a}(t) = a_T \vec{T}(t) + a_N \vec{N}(t)$

$$a_T = \frac{d}{dt}\|v\| = \vec{a}\cdot\vec{T} = \frac{\vec{V}\cdot\vec{a}}{\|\vec{v}\|}$$

$$a_N = \|v\|\cdot\|T'\| = \vec{a}\cdot\vec{N} = \frac{\|\vec{v}\times\vec{a}\|}{\|\vec{V}\|} = \sqrt{\|a\|^2 - a_T^2}$$

Pf

$$\vec{v} = \|\vec{v}\| \frac{\vec{V}}{\|\vec{v}\|} = \|\vec{v}\|\ \vec{T}$$

$$\vec{a}(t) = \vec{V}'(t) = \left(\frac{d}{dt}\|\vec{V}(t)\|\right)\vec{T} + \|\vec{v}\|\ \vec{T}'(t)$$

$$= \left(\frac{d}{dt}\|\vec{v}(t)\|\right)\vec{T} + \|\vec{v}\|\ \|T'(t)\|\cdot\frac{T'(t)}{\|T'(t)\|}$$

$$= \left(\frac{d}{dt}\|\vec{V}(t)\|\right)\vec{T}(t) + \left(\|\vec{V}\|\ \|T'(t)\|\right)\cdot\vec{N}(t)$$

Ex

$r(t) = 3t\vec{i} - t\vec{j} + t^2\vec{k}$

$v(t) = 3\vec{i} - \vec{j} + 2t\vec{k}$, $\|\vec{V}(t)\| = \sqrt{10+4t^2}$

$a(t) = 2k$

$$a_T = \frac{\vec{V}\cdot\vec{a}}{\|V\|} = \frac{4t}{\sqrt{10+4t^2}}$$

$$a_N = \frac{2\sqrt{10}}{\sqrt{10+4t^2}} \qquad \vec{v}\times\vec{a} = -2\vec{i} - 6\vec{j}$$

Ex

$$\vec{T}(t) = \frac{3\vec{i} - \vec{j} + 2t\vec{k}}{\sqrt{10+4t^2}}$$

$\vec{N}(t) =$

$$\vec{r}(t) = b\cos t\,\vec{i} + b\sin t\,\vec{j} + ct\,\vec{k},$$

$$\vec{v}(t) = \vec{r}\,'(t) = -b\sin t\,\vec{i} + b\cos t\,\vec{j} + c\,\vec{k} \qquad b > 0$$

$$\cancel{T(t)}\ \|\vec{v}\| = \sqrt{b^2+c^2}$$

$$\vec{T}(t) = \frac{-b\sin t\,\vec{i} + b\cos t\,\vec{j} + c\,\vec{k}}{\sqrt{b^2+c^2}}$$

$$\cancel{a(t)}\ \vec{a}(t) = \vec{v}\,'(t) = -b\cos t\,\vec{i} - b\sin t\,\vec{j}$$

$$a_T = \frac{\vec{v}\cdot\vec{a}}{\|v\|} = 0$$

$$a_N = \sqrt{\|a\|^2 - a_T^2} = b$$

12.5

$r(t) = x'(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$

$d\vec{r}(t) = dx(t)\,\vec{i} + dy(t)\,\vec{j} + dz\,\vec{k}$

$= x'(t)\,dt\,\vec{i} + y'(t)\,dt\,\vec{j} + z'(t)\,dt\,\vec{k}$

$= (x'(t)\,\vec{i} + y'(t)\,\vec{j} + z'(t)\,\vec{k})\,dt$

$ds = \| d\vec{r}(t) \| = \sqrt{(x'(t)\,dt)^2 + (y'(t)\,dt)^2 + (z'(t)\,dt)^2}$

$= \sqrt{x'^2 + y'^2 + z'^2}\,dt \qquad = \| r'(t) \|\,dt$

$s = \int_a^b ds = \int_a^b \sqrt{x'^2 + y'^2 + z'^2}\,dt = \int_a^b \| d\vec{r} \|$

$= \int_a^b \| r'(t) \|\,dt$

Ex $r(t) = b\cos t\,\vec{i}$
$b\sin t\,\vec{j}$
$\sqrt{1-b^2}\,t\,\vec{k}$

$\int_0^{2\pi} \quad = 2\pi$

Def $\vec{r}(t) : [a, b] \longmapsto \mathbb{R}^3$

$s(t) = \int_a^t \| r'(u) \|\,du = \int_a^t \sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2}\,du$

arc length function

s : arc length parameter

$\frac{ds}{dt} = \| r'(t) \|$, $ds = \| r'(t) \|\,dt$

Ex $r(t) = (3-3t)\vec{i} + 4t\vec{j}$ $\quad 0 \le t \le 1$

$\Rightarrow s(t) = 5t$

$\Rightarrow r(s) = (3 - \frac{3}{5}s)\vec{i} + (\frac{4}{5}s)\vec{j}$

Ex $r(t) = 7\cos t\,\vec{i} + 7\sin t\,\vec{j}$ $\quad t = [0, 2\pi]$

$r'(t) = -7\sin t\,\vec{i} + 7\cos t\,\vec{j}$

$s(t) = \int_0^t 7\,du = 7t$

$\Rightarrow t = \frac{s}{7}$

$r(s) = 7\cos\frac{s}{7}\,\vec{i} + 7\sin\frac{s}{7}\,\vec{j}$

Thm $\vec{r}(s) = x(s)\vec{i} + y(s)\vec{j} + z(s)\vec{k}$

$\Rightarrow \left\|\frac{d}{ds}\vec{r}(s)\right\| = 1$

Ex

Ex

Def C : smooth curve

$\kappa = \left\|\frac{dT(s)}{ds}\right\| = \|T'(s)\|$ curvature

$\underline{\underline{Ex}}$ $\vec{r}(\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j}$

$$r(s) = r\cos\frac{s}{r}\,\vec{i} + r\sin\frac{s}{r}\,\vec{j}$$

$$r'(s) = -\sin\frac{s}{r}\,\vec{i} + \cos\frac{s}{r}\,\vec{j}$$

$$T(s) = \cancel{\frac{\frac{1}{r}\cos\frac{s}{r}\,\vec{i} + \frac{1}{r}(-\sin\frac{s}{r})\,\vec{j}}{\| \quad \|}}$$

$$= \cancel{\cos} = -\sin\frac{s}{r}\,\vec{i} + \cos\frac{s}{r}\,\vec{j}$$

$$T'(s) = -\frac{1}{r}\cos\frac{s}{r}\,\vec{i} - \frac{1}{r}\sin\frac{s}{r}\,\vec{j}$$

$$k = \|T'(s)\| = \frac{1}{r}$$

$\underline{Thm}$

$$k = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|r'(t)\|^3}$$

Moreover $\vec{r} = x\,\vec{i} + y\,\vec{i} = x\,\vec{i} + f(x)\,\vec{j}$

$$\Rightarrow k = \frac{\|y''\|}{[1 + y'^2]^{3/2}}$$

$\underline{Ex}$ $y = x - \frac{1}{4}x^2$ at $x = 2$.

Thm
$$a(t) = \frac{d^2 s}{dt^2}\vec{T}(t) + k\left(\frac{ds}{dt^2}\right)^2 \vec{N}(t)$$

Ex
$$\vec{r}(t) = 2t\,\vec{i} + t^2\,\vec{j} - \frac{1}{3}t^3\,\vec{k}$$

$$\vec{r}'(t) = 2\,\vec{i} + 2t\,\vec{j} - t^2\,\vec{k}$$

$$\frac{ds}{dt} = \|\vec{r}'(t)\| = \sqrt{4 + 4t^2 + t^4} = t^2 + 2.$$

$$T(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{2\,\vec{i} + 2t\,\vec{j} - t^2\,\vec{k}}{t^2+2}$$

$$T'(t) = \frac{-4t\,\vec{i} + (4-2t^2)\,\vec{j} - 4t\,\vec{k}}{(t^2+2)^2}$$

$$\|T'(t)\| = \frac{2}{t^2+2}$$

$$k = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{2}{(t^2+2)^2}$$

$$a_T = \frac{d^2 s}{dt^2} = 2t$$

$$a_N = k\left(\frac{ds}{dt}\right)^2 = \frac{2}{(t^2+2)^2}(t^2+2)^2 = 2$$