11.3

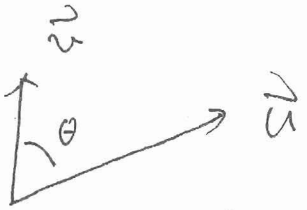
$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$

$\vec{u} = (u_1, u_2, u_3)$

$\vec{v} = (v_1, v_2, v_3)$

property: $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{w}$

Thm



$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$$

pf cosine law $\|\vec{v}-\vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\cdot\|\vec{v}\|\cos\alpha$

def $\vec{u}$ orthorgonal to $\vec{v}$ if $\vec{u}\cdot\vec{v} = 0$

def $\vec{u}, \vec{v}$ nonzero

$\Rightarrow \vec{u} = \vec{w}_1 + \vec{w}_2$

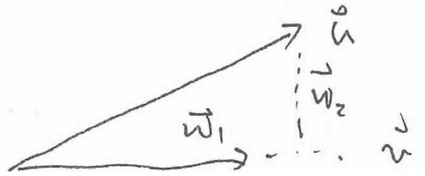
with $\vec{w}_1 \parallel \vec{v}$, $\vec{w}_2 \perp \vec{v}$

(i) $\vec{w}_1$ = the projection of $\vec{u}$ onto $\vec{v}$

= the vector component of $\vec{u}$ along $\vec{v}$

$= \text{proj}_{\vec{v}}\vec{u}$

(ii) $\vec{w}_2$ = vector component of $\vec{u}$ orthognal to $\vec{v}$



then $\vec{w}_1 = \text{proj}_{\vec{v}} \vec{u}$

thus $\text{proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} \right) \vec{v}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2 v_3 - u_3 v_2)\,\vec{i} - (u_1 v_3 - u_3 v_1)\,\vec{j} + (u_1 v_2 - u_2 v_1)\,\vec{k}$$

properties : (i) $\vec{u} \times \vec{v} = - \vec{v} \times \vec{w}$

(ii) $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

Thm : (i) $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0 = (\vec{u} \times \vec{v}) \cdot \vec{u}$

(ii) $\| \vec{u} \times \vec{v} \| = \| \vec{u} \| \cdot \| \vec{v} \| \sin\theta$ = area of ~~parallelo~~ parallelogram with adjacent sides $\vec{u}$ & $\vec{v}$

(iii) $\vec{u} \times \vec{v} = 0 \iff \vec{u} \parallel \vec{v}$

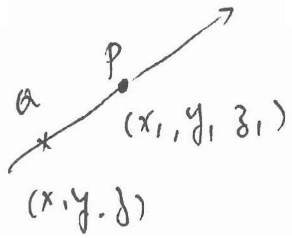
Thm The volume of a parallelepiped with adjacent sides $\vec{u}$, $\vec{v}$, $\vec{w}$ is

$$V = | u \cdot (v \times w) |$$

11.5

line : $P(x_1, y_1, z_1)$

$\vec{V} = (a, b, c)$ direction vector

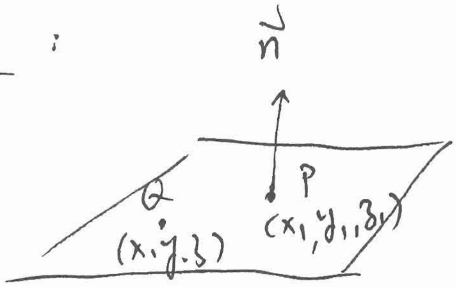


$$\overrightarrow{PQ} = (x - x_1, y - y_1, z - z_1) = t\vec{v}$$

$$\Rightarrow x = at + x_1 \quad y = bt + y_1 \quad z = ct + z_1$$

or $$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

plane :



$\vec{n} = (a, b, c)$ normal vector

$$\overrightarrow{PQ} \perp \vec{n} \Leftrightarrow \overrightarrow{PQ} \cdot \vec{n} = 0$$

$$\Leftrightarrow (x - x_1, y - y_1, z - z_1) \cdot (a, b, c) = 0$$

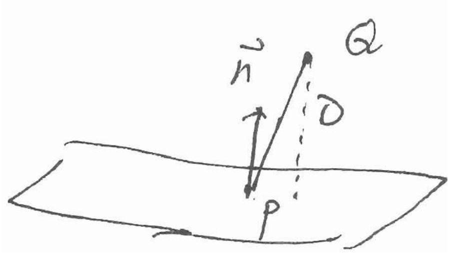
$$\Leftrightarrow a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Ex angle between two plane.

$x - 2y + z = 0$

$2x + 3y - 2z = 0$

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \, \|\vec{n}_2\|}$$



$$\vec{D} = \frac{\vec{n} \cdot \vec{PQ}}{\|\vec{n}\|}$$

- Distance between ~~a point~~ two parallel planes

$$3x - y + 2z - 6 = 0, \quad 6x - 2y + 4z + 4 = 0$$

$$(x_0, y_0, z_0) = (2, 0, 0)$$

- Distane between a point and a line.

11.6

(I) Cylinder $= \{ \text{lines}, \exists L' : L' \parallel L^{\circ} \ \& \ L' \cap C \neq \phi \}$

C : generating curve (or directrix) (a curve in a plane)

L : ruling

Ex (i) $z = y^2$

generating curve $C = z = y^2, \ x = 0$

$L' = x$-axis

(ii) $x^2 + y^2 = a^2$

$C : x^2 + y^2 = a^2, \ z = 0$

L' : z-axis

(iii) $z = \sin x \qquad 0 \le x \le 2\pi$

$C = z = \sin x, \ y = 0$

L' : y-axis

(II) ~~quadrai~~ quadric surface :

$$Ax^2 + Bxy + Cy^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + J = 0$$

(IV) surface of revolution

about x-axis $=$ $y^2+z^2 = r(x)^2$

y $\quad x^2+z^2 = r(y)^2$.

z $\quad x^2+y^2 = r(z)^2 = 0$

Ex

$y = \frac{1}{z}$ about z-axis

$\Rightarrow x^2+y^2 = r(z)^2 = \left(\frac{1}{z}\right)^2$

Cylinder Coordinate System: $P\ (r, \theta, z)$

(r, θ) is a polar coordination of the projection of P in the xy-plane, $(x, y, 0)$

z = the ~~p or~~ directed distance from $(r, \theta, 0)$ to P.

$(x, y, z) \longrightarrow (r, \theta, z)$

$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x}$.

$z = z$

$(r, \theta, z) \longrightarrow (x, y, z)$

$\cancel{r^2 = x^2 + y^2}$ $x = r\cos\theta$

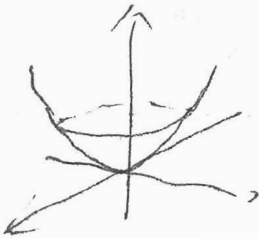
$y = r\sin\theta$.

$z = z$.

Ex (i) $\theta = c$.

(ii) $x^2 + y^2 = 4z$.

$r = 2\sqrt{z}$



Ex ① $x^2+y^2=4z^2 \longrightarrow r^2=4z^2$

$y^2=x \longrightarrow r=\csc\theta\cot\theta$

③ $r^2\cos 2\theta+z^2+1=0$

$\longrightarrow y^2-x^2-z^2=1$

(II) Spherical Coordinates :

a point P in space is discribed by an ordered triple.
(ρ, θ, ϕ)

ρ: distance between P and the origin. $\rho \geq 0$

θ : same angle used in cylindrical coordinates ~~for~~ for $r \geq 0$

ϕ : angle between the positive z-axis and the line $\overline{OP}$.

$0 \leq \phi \leq \pi$.

$(x, y, z) \longrightarrow (\rho, \theta, \phi)$

$\rho^2=x^2+y^2+z^2$

$\tan\theta = \frac{y}{x}$

$\phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

~~Ex~~: $(\rho, \theta, \phi) \longrightarrow (x, y, z)$

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

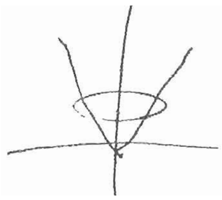
$(\rho, \theta, \phi) \longrightarrow (r, \theta, z)$, $r \geq 0$

$$r^2 = \rho^2 \sin\phi$$
$$\theta = \theta$$
$$z = \rho \cos\phi$$

$(r, \theta, z) \longrightarrow (\rho, \theta, \phi)$

$$\rho = \sqrt{r^2 + z^2}$$
$$\theta = \theta$$
$$\phi = \arccos\left(\frac{z}{r^2 + z^2}\right)$$

Ex
$\rho = \text{const}$
$\theta = \text{const}$
$\phi = \text{const}$



$\phi = c.$

Ex ① $x^2 + y^2 = z^2 \longrightarrow \phi = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

② $x^2 + y^2 + z^2 - 4z = 0 \longrightarrow \rho(\rho - 4\cos\phi) = 0$

$\longrightarrow \rho = 4\cos\phi$