9.1

sequence : $\mathbb{N} = \{1, 2, 3 \dots\} \longrightarrow \mathbb{R}$

$$\begin{array}{ccc} 1 & \longrightarrow & a_1 \\ 2 & \longrightarrow & a_2 \\ \vdots & & \vdots \\ n & \longrightarrow & a_n \end{array}$$

given by explicitly or by formula.

ex $\{a_n\} = \{3 + (-1)^n\}$

$a_1 = 2$, $a_2 = 4$, $a_3 = 2$, $a_4 = 4$, ...

ex $\{b_n\} = \left\{ \dfrac{n}{1-2n} \right\}$

$b_1 = -1$, $b_2 = -\frac{2}{3}$, $b_3 = -\frac{3}{5}$, $b_4 = -\frac{4}{7}$, ...

[illegible] sometimes, recursively defined

① $d_1 = 25$

$d_{n+1} = d_n - 5$

$\Rightarrow d_1 = 25$, $d_2 = 20$, $d_3 = 15$

② $e_1 = 1$, $e_2 = 1$

$e_{n+2} = e_{n+1} + e_n$

$\Rightarrow$ [illegible] 1. 1, 2, 3, 5, 8

(

Def : Limit of $\{a_n\}$, $\lim_{n\to\infty} a_n = L$

~~lim~~ $\forall \varepsilon > 0 \quad \exists M > 0$

$|a_n - L| < \varepsilon$ for $n > M$

Thm : ~~an a function f~~ $\lim_{x\to\infty} f(x) = L$

$a_n = f(n)$

$\Rightarrow \lim_{n\to\infty} a_n = L$

ex $a_n = (1+\frac{1}{n})^n$

$f(x) = (1+\frac{1}{x})^x \qquad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} (1+\frac{1}{x})^x = e$

$\lim_{n\to\infty} a_n = e$

Thm : $\lim_{n\to\infty} a_n = L \qquad \lim_{n\to\infty} b_n = M$

$\Rightarrow$ ① $\lim_{n\to\infty}(a_n \pm b_n) = L + M$

② $\lim_{n\to\infty}(a_n \cdot b_n) = L \cdot M$

③ $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ for $M \neq 0$, $b_n \neq 0$

ex $a_n = \frac{n^2}{2^n - 1}$

$f(x) = \frac{x^2}{2^x - 1} \qquad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{2^x - 1} = \lim_{x\to\infty} \frac{2x}{(\ln 2) 2^x}$

$= \lim_{x\to\infty} \frac{2}{(\ln 2)(\ln 2) 2^x} = 0$

$\Rightarrow \lim_{n\to\infty} a_n = 0$

Thm $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$

$$a_n \le c_n \le b_n$$

$$\Rightarrow \lim_{n\to\infty} c_n = 0$$

ex $c_n = (-1)^n \frac{1}{n!}$

$$a_n = -\frac{1}{n!} \qquad b_n = \frac{1}{n!} \qquad \Rightarrow \quad a_n \le c_n \le b_n$$

$$\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$$

$$\Rightarrow \lim_{n\to\infty} c_n = 0$$

~~Def~~ Thm: $\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$

pf

$$-|a_n| \le a_n \le |a_n|$$

Def $\{a_n\}$ is if a_n are nondecreasing

$$a_1 \le a_2 \le a_3 \cdots$$

a_n are nonincreasing

$$a_1 \ge a_2 \ge a_3 \ge \cdots$$

Def a_n bounded above if $a_n \le M \ \forall n$, M upper bound

bounded below if $a_n \ge N \ \forall n$, N lower bound

bounded if bounded above and bounded below

Thm 9.5

$\{a_n\}$ bounded, monotonic

$\Rightarrow \{a_n\}$ convergent

ex $\{a_n\} = \{\frac{1}{n}\}$

$0 \le a_n < 2 \Rightarrow a_n$ bounded

$a_1 > a_2 > \cdots \Rightarrow a_n$ non increasing $\Rightarrow a_n$ monotonic

$\Rightarrow a_n$ convergent

in fact, $\lim_{n\to\infty} a_n = \frac{1}{n}$

ex $\{c_n\} = \{(-)^n\}$

$-1 \le c_n \le 1 \Rightarrow c_n$ bounded

c_n ~~non~~ $\Rightarrow$ not non increasing, not non decreasing $\Rightarrow$ not monotonic.

9.2

Def $\{a_n\}$ an infinite sequence,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots \quad \underline{\text{infinite series}} \text{ or } \underline{\text{series}}$$

Def: $S_n = a_1 + a_2 + \cdots + a_n$ <u>nth partial sum</u>

if $\lim_{n\to\infty} S_n = S$, $\sum_{n=1}^{\infty} a_n$ <u>converges</u> and

$$S = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

if S_n diverges, then the series <u>diverges</u>

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$\vdots$

$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$

$\lim_{n\to\infty} S_n = 1$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$S_1 = \frac{1}{1\cdot 2} = \frac{1}{2} = 1 - \frac{1}{2}$

$S_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3}$

$S_3 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = 1 - \frac{1}{4}$

$S_n = \cdots \frac{1}{\ }$ $\quad \Rightarrow \lim S_n = 1$

ex $\sum_{n=1}^{\infty} 1$

$S_1 = 1,\ S_2 = 2,\ S_3 = 3,\ \cdots\ S_n = n\quad \lim S_n = \infty$

ex $\sum_{n=1}^{\infty} \frac{2}{4n^2-1}$

$$a_n = \frac{2}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{1}{2n-1} - \frac{1}{2n+1}$$

$$S_n = 1 - \frac{1}{2n+1}$$

$$\lim_{n\to\infty} S_n = 1$$

Def : $\sum_{n=0}^{\infty} a\cdot r^n = a + ar^1 + ar^2 + \cdots + ar^n + \cdots,\quad a \neq 0$

geometric series

Def : $$\sum_{n=0}^{\infty} ar^n = \begin{cases} \frac{a}{1-r} & \cancel{0}<|r|<1 \\ \text{divergent} & |r| \geq 1 \end{cases}$$

pf : $S_n = a + ar + \cdots + ar^{n-1}$

$rS_n = \quad ar + \cdots + ar^{n-1} + ar^n$

$$(1-r)S_n = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

① $\cancel{0}<|r|<1\quad \lim_{n\to\infty} S_n = \frac{a}{1-r}$

② ~~by Thm 9.8~~ $|r| \geq 1\quad \lim_{n\to\infty} |a_n| = \lim_{n\to\infty} a|r^n| \neq 0$

by Thm 9.8 divergent

nth-Term Test

Thm $\sum_{n=1}^{\infty} a_n$ convergent $\Rightarrow \lim_{n\to\infty} a_n = 0$

($\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ divergent)

If $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = L$

$$S_n = S_{n-1} + a_n$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (S_{n-1} + a_n) = \lim_{n\to\infty} S_{n-1} + \lim_{n\to\infty} a_n$$

$$\Rightarrow L = L + \lim_{n\to\infty} a_n$$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

9.3

$f(x)$: positive, continuous, decreasing

$a_n = f(n)$

$\Rightarrow \sum_{n=1}^{\infty} a_n$, $\int_1^{\infty} f(x)\,dx$ <u>either</u> both converge <u>or</u> both diverge.

pf $\sum_{i=2}^{n} f(i) \le \int_1^n f(x) \le \sum_{i=1}^{n} f(i)$

$S_n - a_1 \le \int_1^n f(x) \le S_n$

$S_n = a_1 + \cdots + a_n$
$= S_{n-1} + f(n)$, $f(n) \ge 0$
$\Rightarrow S_n \ge S_{n-1}$
$\Rightarrow S_n$ increasing

(I) ① $\int_1^{\infty} f(x)\,dx = \lim_{n\to\infty} \int_1^n f(x)\,dx = L$ convergent

$S_n - a_1 \le L \Rightarrow S_n \le L + a_1$ bounded

S_n convergent

② $\sum_{n=1}^{\infty} S_n = S$ convergent

$\int_1^n f(x)\,dx \le S_n < S$

$\int_1^{n} f(x)\,dx$ monotonic, bounded $\Rightarrow \int_1^{\infty} f(x)\,dx$ convergent

(II) ① $\int_1^n f(x)\,dx$ divergent

$S_n \ge \int_1^n f(x)\,dx$ divergent

② S_n divergent

$\int_1^n f(x)\,dx \ge S_n - a_1$ divergent ex $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$f(x) = \frac{x}{x^2+1}$ positive, $f'(x) = \frac{-x^2+1}{x^2+1} < 0, \ x > 1$

decreasing

$\lim_{n\to\infty} \int_1^n f(x)\,dx = \lim_{n\to\infty} \int_1^n \frac{x}{x^2+1}\,dx$

$= \lim_{n\to\infty} \frac{1}{2} \cdot \ln(x^2+1) \Big|_1^n$

$= \frac{1}{2} \lim_{n\to\infty} \ln(b^2+1) - \frac{1}{2}\ln 2 \longrightarrow \infty$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n}{n+1}$ divergent.

ex $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$f(x) = \frac{1}{x^2+1} > 0$

$f'(x) = \frac{-2x}{(x^2+1)^2} < 0$

$\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{n\to\infty} \int_1^n \frac{dx}{x^2+1} = \lim_{n\to\infty} \arctan x \Big|_1^n$

$= \lim_{n\to\infty} (\arctan n - \arctan 1)$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ convergent

Def $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ <u>p-series</u>

$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$ <u>harmonic</u> series

$\sum_{n=1}^{\infty} \frac{1}{an+b} = \frac{1}{a+b} + \frac{1}{2a+b} + \cdots$ <u>general harmonic series</u>

<u>Thus</u> $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \cdots = \begin{cases} \text{converge} & 1 < p \\ \text{divergen} & 0 < p \le 1 \end{cases}$

<u>ex</u> $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ divergent

Thm $0 < a_n \le b_n$

① $\sum_{n=1}^{\infty} b_n$ convergent $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ convergent

② $\sum_{n=1}^{n} a_n$ divergent $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ divergent.

ex $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

$$\frac{1}{2+3^n} < \frac{1}{3^n}$$

ex $\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$ $\left(\frac{1}{2+\sqrt{n}} < \frac{1}{\sqrt{n}} \right.$, fails useless $\left.\right)$

$$\frac{1}{n} \le \frac{1}{2+\sqrt{n}} \qquad n \ge k$$

Thm $a_n > 0$, $b_n > 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0,$$

$\Rightarrow$ $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ either both converge

or both divergent

ex $\sum_{n=1}^{\infty} \frac{1}{an+b}$ diverges ($a>0$. $b>0$)

ex $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges

ex $\displaystyle\sum_{n=1}^{\infty} \frac{n\,2^n}{4n^3+1}$ diverges

since $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges.

9-5 $\sum_{n=1}^{\infty}(-)^n a_n$, or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$, $a_n \ge 0$

Def $\sum_{n=1}^{\infty}(-1)^n a_n$ alternating series if

$a_n = (-1)^n b_n$ ... $b_n \ge 0$

or $a_n = (-1)^{n+1} c_n$ ... $c_n \ge 0$

Thm $a_n > 0$.

$\sum_{n=1}^{\infty}(-1)^n a_n$, $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$

convergent if

① $\lim_{n\to\infty} a_n = 0$

② $a_{n+1} \le a_n$ $\forall n$.

Pf consider $\sum_{1}^{\infty}(-1)^{n+1} a_n$

$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{n-1} - a_n)$

mononotonic (nondecreasing)

$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - a_{2n} \le a_1$

bounded

$\Rightarrow \lim_{n\to\infty} S_{2n} = L$

$\lim_{n\to\infty} S_{2n-1} = \lim_{n\to\infty}(S_{2n} + a_{2n}) = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n} = L + 0 = L$

$\Rightarrow \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = L$

ex $\sum (-)^{n+1} \frac{1}{n}$ convergent ex $\sum_{n=1}^{\infty} \frac{n}{(-2)^{n-1}}$ converges

Ex : (thm fails.)

$\sum_{n=1}^{\infty} (-)^{n+1} \frac{n+1}{n}$

$\lim_{n\to\infty} \frac{n+1}{n} = 1$ $\Rightarrow$ Thus Thm can not apply.

Thm $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ convergent

$a_{n+1} \le a_n$, $S_n = a_1 - a_2 + \cdots + (-)^{n+1} a_n$

$R_N = S - S_n$

$\Rightarrow |R_N| \le a_{N+1}$

pf $R_N = (-1)^N (a_{N+1} - a_{N+2} + a_{N+3} + \cdots)$

$= (-1)^N [a_{N+1} - (a_{N+2} - a_{N+3}) - (a_{N+4} - a_{N+5}) - \cdots]$

$|R_N| \le a_{N+1}$

ex $\sum_{n=1}^{\infty} (-)^{n+1} \frac{1}{n+1}$

$|S_6 - S| = |R_6| \le a_7 = \frac{1}{7!} = \frac{1}{5040} \approx 0.0002$

$S_6 = \frac{1}{1} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} = \frac{91}{144} \approx 0.6319$

$S_6 - a_7 \le S \le S_6 + a_7$

Thm $\Sigma|a_n|$ converge $\Rightarrow$ Σa_n converge

pf

$$0 \le a_n + |a_n| \le 2|a_n|$$

$\Longrightarrow$ $\Sigma a_n + |a_n|$ convegent

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n| \quad \text{convegent.}$$

ex $\sum_{n=1}^{\infty} (-)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$ conv

but $\sum_{n=1}^{\infty} \left|(-)^{n+1}\frac{1}{n}\right| = 1 + \frac{1}{2} + \cdots$ divergent

Def Σa_n <u>absolutely convergent</u> if $\Sigma|a_n|$ convergent

<u>conditional convergent</u> if $\Sigma|a_n|$ divergent ~~but~~ but

Σa_n convergent

ex $\sum_{n=1}^{\infty} \frac{(-)^n}{\sqrt{n}}$ conditional convergent

$\sum_{n=1}^{\infty} \frac{(-)^n n!}{2^n}$ divergent (since $\lim_{n\to\infty} \frac{(-)^n n!}{2^n} \neq 0$)

$\sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n+1)}{2}}}{3^n}$ absolutely convergent

$\sum_{n=1}^{\infty} \frac{(-)^n}{\ln(n+1)}$ conditional convergent

9.6

Thm $\sum a_n$, $a_n \neq 0$

~~1. converge absolutely if~~ $\cancel{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1}$

~~2. div~~

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \sum a_n \text{ converge absolutely}$$

$$\geq 1 \text{ or } \infty \Rightarrow \sum a_n \text{ diverges}$$

$$= 1 \Rightarrow \text{unknown.}$$

Ex $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ <u>convergent</u>

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2}{n+1} \longrightarrow 0$$

Ex $\sum_{n=0}^{\infty} \frac{n^2\, 2^{n+1}}{3^n}$ <u>convergent</u>

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2\, 2^{n+2}}{3^{n+1}}}{\frac{n^2\, 2^{n+1}}{3^n}} = \frac{(n+1)^2}{n^2}\,\frac{2}{3} \longrightarrow \frac{2}{3}$$

Ex $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ <u>divergent</u>

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \frac{(n+1)^{n+1}}{(n+1)\, n^n} = \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)^n \longrightarrow e$$

Ex $\sum_{n=1}^{\infty} (-)^n \frac{\sqrt{n}}{n+1}$

① $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{n+2}}{\frac{\sqrt{n}}{n+1}} = \lim \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \lim \frac{n+1}{n+2} = 1$

$\Rightarrow$ Ration Test fails

② Alternating Series test

$f(x) = \frac{\sqrt{x}}{x+1}$

$f'(x) = \frac{-x+1}{2\sqrt{x}(x+1)^2} < 0 \quad , x<1$

$\Rightarrow$ $f'(x)$ decreasing

$a_{n+1} = f(n+1) \leq f(n) = a_n$

$\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$

$\Rightarrow \sum_{n=1}^{\infty} (-)^n \frac{\sqrt{n}}{n+1}$ convergent

But $\sum_{n=1}^{\infty} \left| (-)^n \frac{\sqrt{n}}{n+1} \right| = \sum_{n=1} \frac{\sqrt{n}}{n+1}$ divergent

therefor

$\sum_{n=1}^{\infty} (-)^n \frac{\sqrt{n}}{n+1}$ conditionally convergent

Thm :

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} \quad < 1 \quad \Rightarrow \sum_{n=1}^{\infty} a_n \quad \text{convergent absolutely}$$

$$\left\{\begin{array}{l} > 1 \\ = \infty \end{array}\right. \quad \Rightarrow \sum_{n=1}^{\infty} a_n \quad \text{divergent}$$

$$= 1 \quad \Rightarrow \sum_{n=1}^{\infty} a_n \quad \text{unknown}$$

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

$$\left|\frac{e^{2n}}{n^n}\right|^{\frac{1}{n}} = \frac{e^2}{n} \longrightarrow 0 < 1$$

(all together)

Ex $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{3n+1}$

$$\sum_{n=1}^{\infty} \left(\frac{\pi}{6}\right)^n$$

$$\sum_{n=1}^{\infty} n e^{-n^2}$$

$$\sum_{n=1}^{\infty} (-)^n \frac{3}{4n+1}$$

$$\sum_{n=1}^{\infty} \frac{n!}{10^n}$$

$$\sum \left(\frac{n+1}{2n+1}\right)^n$$

9.7

Given : $f(x)$

Goal: To find a polynomial function $P(x)$

$$P(x) = a_0 + a_1 \cancel{x} \cdots + a_n (x - c)^n$$

that approximate $f(x)$ such that $\widehat{\text{at}}$ $x = c$

$$P(c) = f(c)$$

$$P'(c) = f'(c)$$

$$\vdots$$

$$P^{(n)}(c) = f^{(n)}(c)$$

$\Rightarrow$

$$a_0 = f(c)$$

$$a_1 = f'(c)$$

$$a_2 = \frac{1}{2!} f''(c)$$

$$a_n = \frac{1}{n!} f^{(n)}(c)$$

Def f has n derivative at c,

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{1}{2!} f''(c)(x-c)^2 + \cdots + \frac{1}{n!} f^{(n)}(c)(x-c)^n$$

<u>nth Taylor polynomial for f at $x = c$.</u>

If $c = 0$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n$$

<u>nth Maclaurin polynomial for f</u>.

$\underline{\underline{Ex}}$ $f(x) = e^x$

$f'(x) = e^x$

$f^{(n)} = e^x$

$c = 0$ $\quad f(0) = 1 = f^{(1)}(0) = \cdots = f^{(n)}(0)$

$$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$$

$\underline{\underline{Ex}}$ $f(x) = \ln x$ at $c = 1$

$f(x) = \ln x$ $\qquad f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x} = x^{-1}$ $\qquad f'(1) = 1$

$f''(x) = (-)x^{-2}$ $\qquad f''(1) = -1$

$f'''(x) = (-1)(-2)x^{-3}$ $\qquad f'''(1) = 2$

$f^{(4)}(x) = (-1)(-2)(-3)x^{-4}$ $\qquad f^{(4)}(1) = -6$

$P_0(x) = 0$

$P_1(x) = f(1) + \frac{f'(1)}{1!}(x-1) = x - 1$

$P_2(x) = f(1) + \frac{f'(1)(x-1)}{1!} + \frac{f''(1)}{2!}(x-1)^2 = (x-1) + \frac{-1}{2!}(x-1)^2$

$P_3(x) = (x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$

$P_4(x) = (x-1) + \frac{-1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 + \frac{-6}{4!}(x-1)^4$

ex $f(x) = \cos x$ $\quad$ $c = 0$

$P_0(x) = 1$

$P_2(x) = 1 - \frac{1}{2!}x^2$

$P_4(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$

$P_6(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$

ex $f(x) = \sin x$ $\quad$ $c = \frac{\pi}{6}$

$$P_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{2(2!)}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{2(3!)}\left(x - \frac{\pi}{6}\right)^3$$

Thm $f(x) = \underbrace{P_n(x)}_{\text{Approximation value}} + \underbrace{R_n(x)}_{\text{remainder}}$

$|R_n| = |f(x) - P_n(x)|$ error

Taylay's Thm

Thm : f differentiable of order $n+1$ in I

$c \in I$,

$\Rightarrow$ there exists a z between x and c.

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(x)}{n!}(x-c)^n$$

$+ R_n(x)$

with $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$ $\quad$ Langrange form of th ex $\quad f(x) = \sin x \qquad c=0, \quad x=0.1$

$$P_3(x) = x - \frac{x^3}{3!}$$

$$\Rightarrow f(x) = P_3(x) + R_3(x)$$

i.e. $\sin x = x - \frac{x^3}{3!} + \frac{f^{(4)}(z)}{4!} x^4$

$$f^{(4)}(z) = \sin z$$

$$|R_3(0.1)| = \frac{\sin z}{4!}(0.1)^4 < \frac{\sin(0.1)}{4!}(0.1)^4$$

$$< \frac{0.0001}{4!} \qquad (\sin 0.1 < 1)$$

$$P_3(0.1) = 0.1 - \frac{(0.1)^3}{3!} = 0.1 - \frac{0.001}{6}$$

$$= 0.1 - 0.000167$$
$$= 0.099833$$

$$\Rightarrow P_3(0.1) - \frac{0.0001}{4!} < \sin 0.1 = P_3(0.1) + R_3(x) = P_3(0.1) + \frac{0.0001}{4!}$$

ex $\quad \ln 1.2 = ? \qquad$ error < 0.001

$f(x) = \ln x \qquad c=1 \qquad x=1.2$

$$f^{(n+1)}(x) = (-)^n \frac{n!}{x^{n+1}}$$

$$|R_n(1.2)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!}(1.2-1)^{n+1} \right|$$

$$= \frac{1}{n+1}\frac{1}{z^{n+1}}(0.2)^{n+1}$$

9-7-X

with $1 \le z \le 1.2$

$$\Rightarrow \left| R_n(1.2) \right| < \frac{1}{n+1} \frac{1}{1^{n+1}} (0.2)^{n+1} = \frac{(0.2)^{n+1}}{n+1} < 10^{-3}$$

$\Rightarrow n = 3$

$$P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

$$P_3(1.2) = 0.2 - \frac{1}{2}(0.2)^2 + \frac{1}{3}(0.2)^3$$

9.8

Def : $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \cdots$ a power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

a power series centered at c

Ex
$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots$$

$$\sum_{n=0}^{\infty} (-1)^n (x-1)^n = 1 - (x-1) + (x-1)^2 + \cdots$$

Thm : $\sum_{n=0}^{\infty} a_n (x-c)^n$

① convergent only at $x = c$.

② There is a real number R,

$\sum_{n=0}^{\infty} a_n (x-c)^n$ convergent $|x-c| < R$

divergent $|x-c| > R$

③ convergent absolutely $\forall x \in \mathbb{R}$

R radius of convergence of the power series.

$R = 0$ for the case ①

$R = \infty$ for the case ③

~~$I = (c-R, c+R)$~~ interval of convergence

$I = \{x \mid \sum_{n=0}^{\infty} a_n (x-c)^n \text{ convergent}\}$

Ex $\sum_{n=0}^{\infty} n!\, x^n$

$x=0$ $\sum_{n=0}^{\infty} n!\, 0^n = 1+0+\cdots = 1$

$x \neq 0$ $\sum_{n=0}^{\infty} n!\, x^n$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!\, x^{n+1}}{n!\, x^n} = (n+1)\cdot x \longrightarrow \infty$$

$\sum_{n=0}^{\infty} n!\,x^n$ divergent.

$\Rightarrow \sum_{n=0}^{\infty} n!\, x^n$ convergent only at $x=0$

Ex $\sum_{n=0}^{\infty} 3(x-2)^n$

$u_n = 3(x-2)^n$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{3(x-2)^{n+1}}{3(x-2)^n}\right| = \lim_{n\to\infty} |x-2| = |x-2|$$

$\Rightarrow \sum_{n=0}^{\infty} 3(x-2)^n$ convergent if $|x-2|<1$

divergent if $|x-2|>1$

$R=1$ radius of convergence

~~(1, 3) radius of convergence~~ ~~interval~~

Ex $\displaystyle\sum_{n=0}^{\infty} \frac{(-)^n x^{2n+1}}{(2n+1)!}$

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{(-)^n x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{2n+3}}{(2n+3)!}}{\frac{x^{2n+1}}{(2n+1)!}} \right|$$

$$= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)}$$

$$= 0$$

$\Rightarrow \displaystyle\sum_{n=0}^{\infty} \frac{(-)^n}{(2n+1)!} x^{2n+1}$ convergent $\forall x$.

$R = \infty$

~~Endpoint convergent~~ the endpoints

what happens at $c-R$ and $c+R$?

Ex $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n}$

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \right| \cdot |x| = |x|$$

$\Rightarrow \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n}$ convergent $|x| < 1$

divergent $|x| > 1$

$x = 1$ $\quad \displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$ divergent

$x = -1$ $\quad \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ convergent

$\Rightarrow I = [-1, 1)$ interval of convergence

$\underline{\underline{Ex}}$ $\quad \sum \frac{(-)^n (x+1)^n}{2^n}$

$$\lim_{n\to\infty} \left| \frac{\frac{(-)^{n+1}(x+1)^{n+1}}{2^{n+1}}}{\frac{(-)^n (x+1)^n}{2^n}} \right| = \lim_{n\to\infty} \left| \frac{x+1}{2} \right| = \left| \frac{x+1}{2} \right|$$

$\left|\frac{x+1}{2}\right| < 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$ convegent

$\left|\frac{x+1}{2}\right| > 1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \frac{(-1)(x+1)^n}{2^n}$ divergent

$\cancel{x} \; \frac{x+1}{2} = \pm 1 \Rightarrow x = -3, 1$

$x = -3 \quad \sum_{n=0}^{\infty} \frac{(-)^n (-2)^n}{2^n} = \sum_{n=0}^{\infty} 1$ div.

$x = 1 \quad \sum_{n=0}^{\infty} \frac{(-)^n 2^n}{2^n} = \sum_{n=0}^{\infty} (-1)$ div.

$I = (-3, 1)$ interval of convergence

$\underline{\underline{Ex}}$ $\quad \sum_{n=0}^{\infty} \frac{x^n}{n^2} \qquad I = [-1, 1]$

Thm $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$

$= a_0 + a_1(x-c) + \cdots + a_n(x-c)^n + \cdots$

has radius of convergence R.

$\Rightarrow$ on the interval $(c-R, c+R)$

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots + n a_n (x-c)^{n-1} + \cdots$$

$$\int f(x)\,dx = C + a_0(x-c) + a_1 \frac{(x-c)^2}{2} + \cdots + \frac{a_n}{n+1}(x-c)^{n+1} + \cdots$$

The radius of convergence of the above power series is R.

But the interval of convergence may be difference.

Ex $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \qquad I = [-1, 1)$

$f'(x) = 1 + x + x^2 + \cdots \qquad I = (-1, 1)$

$\int f(x)\,dx = C + \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \cdots, \qquad I = [-1, 1]$

9.9

$f(x) = \frac{1}{1-x}$

We kown that
$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad , |r|<1 ,$$

$a=1 \ , \ r=x$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x|<1$$

$$\frac{1}{1-x} = \frac{1}{2-(x+1)} = \frac{\frac{1}{2}}{1-\left(\frac{x+1}{2}\right)}$$

$a = \frac{1}{2} \ , \ r = \frac{x+1}{2}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{x+1}{2}\right)^n = \frac{1}{2}\left[1 + \left(\frac{x+1}{2}\right) + \left(\frac{x+1}{2}\right)^2 + \left(\frac{x+1}{2}\right)^3 + \cdots\right]$$

for $\left|\frac{x+1}{2}\right| < 1$ i.e. $x \in (-3, 1)$

Thm: $f(x) = \sum a_n x^n \ , \ g(x) = \sum b_n x^n$

$$\Rightarrow \quad f(kx) = \sum_{n=0}^{\infty} a_n (kx)^n$$

$$f(x^k) = \sum_{n=0}^{\infty} a_n x^{kn}$$

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$$

Find a power series centered at 0 for

Ex $\quad f(x) = \frac{3x-1}{x^2-1} \ ,$ ~~x=0~~

$$\frac{3x-1}{x^2-1} = \frac{2}{x+1} + \frac{1}{x-1}$$

$$= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$x = 0$ $\quad g(0) = 0 = C + 0 \quad \Rightarrow \quad C = 0$

$$\Rightarrow g(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad |x| < 1$$

Thm $f(x)$ represented by a power series $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ in an interval I with $c \in I$, then

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!} + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

Def f has deriverative of all order at $x=c$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

<u>Taylor series for $f(x)$ at c</u>

$c = 0$ <u>Maclaurin series for f.</u>

ex $f(x) = \sin x$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$$

Thm If $\lim_{n\to\infty} R_n = \lim_{n\to\infty} f(x) - P_n(x) = 0$,

then Taylor series converges and equal to $f(x)$, i.e.,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Ex $f(x) = \sin x$

$f^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x$

$$|R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right|$$

$$= \frac{|f^{(n+1)}(z)|}{} \cdot \frac{|x|^{n+1}}{(n+1)!}$$

$$\leq \frac{|x|^{n+1}}{(n+1)!}$$

$$\longrightarrow 0$$

$$\Rightarrow \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad x \in \mathbb{R}$$

Ex $f(x) = \sin x^2$

$$\doteq x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots \qquad x \in \mathbb{R}.$$

$$= x^2 - \frac{x^6}{3!} + \frac{x^6}{5!} - \frac{x^{14}}{7!} + \cdots$$

Ex $f(x) = \cos \sqrt{x}$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots \qquad x \in \mathbb{R}$$

$$\cos \sqrt{x} = 1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \frac{(\sqrt{x})^8}{8!} + \cdots$$

$$= 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} + \cdots$$

Ex

$$f(x) = (1+x)^k \qquad f(0)=1$$

$$f'(x) = k(1+x)^{k-1} \qquad f'(0)=k$$

$$f''(x) = k(k-1)(1+x)^{k-2} \qquad f''(0)= k(k-1)$$

$$f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n} \qquad f^{(n)}(0) = k(k-1)\cdots(k-n+1)$$

$$P_n(x) = 1 + kx + \frac{k(k-1)}{2}x^2 + \cdots + \frac{k(k-1)\cdots(k-n+1)}{n!}x^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{k(k-1)\cdots(k-n)}{(n+1)!}x^{n+1}}{\frac{k(k-1)\cdots(k-n+1)}{n!}x^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{k-n}{n+1}\right|\cdot|x|$$

$$= |x|$$

$$\Rightarrow (1+x)^k = 1 + kx + \frac{k(k-1)}{2}x^2 + \cdots \qquad \forall\ |x|<1$$