Def: $F$ antiderivative of $f$ if $F'(x) = f(x)$

Thm: $F$ antiderivative of $f$ (i.e. $F'(x) = f(x)$)

$G$ antiderivative of $f$ iff $G(x) = F(x) + c$.
($G'(x) = f(x)$)

- antiderivative
- $\int f(x)dx = F(x) + C$
- $C = ?$

Pf "$\Leftarrow$" Suppose $G(x) = F(x) + C$

Then $G'(x) = F'(x) = f(x)$

therefor $G(x)$ is an antiderivative of $f(x)$

"$\Rightarrow$" $H(x) = G(x) - F(x)$

$\Rightarrow H'(x) = G'(x) - F'(x) = f(x) - f(x) = 0$

$\forall\, a, b \in I$ by M.V.T

$H(b) - H(a) = H'(c) \cdot (b - a)$ for some $c$

$= 0$

Hence $H(b) = H(a) = \text{constant} = C$

$\Rightarrow G(x) = F(x) + C.$

Ex: $y' = 2$

$\Rightarrow y = 2x + C$

$y' = 3x^2 - 1$

$y = \int (3x^2-1)\,dx = x^3 - x + C$ <u>general sol.</u>

$x=2,\ y=4$ <u>initial condition</u>

$4 = 2^3 - 2 + C \Rightarrow C = -2$

$y = x^3 - x - 2$ <u>particular sol</u>

<u>Ex</u>

$s(t)$

$s(0) = 80$

$s'(0) = 64$

$s''(t) = -32$

$\Rightarrow s'(t) = \int s''(t)\,dt = \int -32\,dt = -32t + C.$

$t=0$, $64 = -32 \cdot 0 + C \Rightarrow C = 64$

$s'(t) = -32t + 64$

$s(t) = \int s'(t)\,dt = \int (-32t + 64)\,dt$

$= -16t^2 + 64t + C$

$t=0$ $80 = -16 \cdot 0^2 + 64 \cdot 0 + C \Rightarrow C = 80$

$\Rightarrow s(t) = -16t^2 + 64t + 80$

<u>Notation</u> :

$$\frac{dy}{dx} = f(x)$$

$$dy = f(x)\,dx$$

$$y = \int dy = \underbrace{\int f(x)\,dx}_{\text{antiderivative of } f \text{ with respect to } x} = F(x) + C.$$

<u>Basic Rule</u> :

(i) $\int F'(x)\,dx = F(x) + C$

(ii) $\frac{d}{dx} \int f(x)\,dx = f(x)$

<u>Ex</u> $\int 3x\,dx = \frac{3}{2} x^2 + C.$

<u>Ex</u> $\int \frac{1}{x^3}\,dx = \frac{1}{-2} x^{-2} + C.$

<u>Ex</u> $\int \frac{x+1}{\sqrt{x}}\,dx = \int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx$

$$= \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

Ex $F'(x) = \frac{1}{x^2}$ , $x > 0$ with $F(1) = 0$

# 4-2

- Notation: $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$

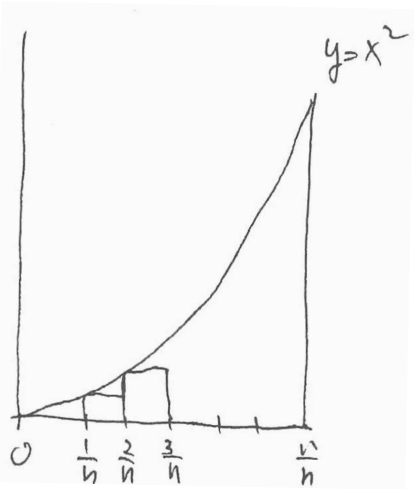
Ex: $\sum_{i=1}^{n} i^2 = 1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Ex $\sum_{i=1}^{n} i^3 = 1 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

Ex $\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x$

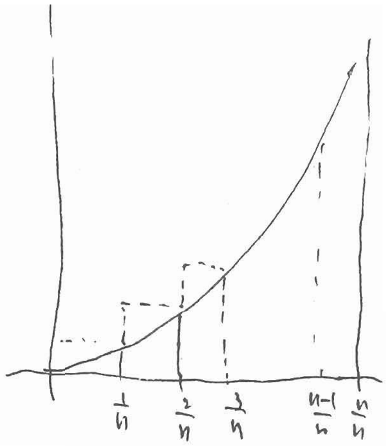
Ex
$$\sum_{i=1}^{n} \frac{i+1}{n^2} = \sum_{i=1}^{n} \frac{i}{n^2} + \sum_{i=1}^{n} \frac{1}{n^2}$$
$$= \frac{1}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^2}\sum_{i=1}^{n} 1$$
$$= \frac{1}{n^2}\cdot\frac{n(n+1)}{2} + \frac{1}{n^2}\cdot n$$
$$= \frac{n+1}{2n} + \frac{1}{n} = \frac{3n+1}{2n}$$

Area




$$R(n) = 0\cdot\left(\frac{1}{n}\right) + \left(\frac{1}{n}\right)^2\cdot\frac{1}{n} + \left(\frac{2}{n}\right)^2\cdot\frac{1}{n} + \cdots + \left(\frac{n-1}{n}\right)^2\cdot\frac{1}{n}$$
$$= \sum_{i=1}^{n-1} \frac{i^2}{n^3}$$

$$r(n) = f(m_1)\Delta x + f(m_2)\Delta x \cdots + f(m_n)\Delta x$$
$$= \sum_{i=1}^{n} f(m_i)\Delta x = \text{lower sum}$$




$$S(n) = \left(\frac{1}{n}\right)^2 \cdot \frac{1}{n} + \left(\frac{1}{n}\right)^2 \frac{1}{n} + \left(\frac{1}{n}\right)^{\frac{2}{n}} \cdot \frac{1}{n} + \cdots + \left(\frac{n}{n}\right)^2 \cdot \frac{1}{n}$$

$$= \sum_{i=1}^{n} \frac{i^2}{n^3}$$

$$S(n) = f(M_1)\Delta x + f(M_2)\Delta x + \cdots + f(M_n)\Delta x$$

$$= \sum_{i=1}^{n} f(M_i)\Delta x$$

$= \text{upper Sum}$

$s(n)$ = lower sum

$S(n)$ = upper sum

$$s(n) \leq A \leq S(n)$$

$\lim s(n) = \frac{1}{3}$

$\lim S(n) = \frac{1}{3}$

$\Rightarrow$ The area is 2.

<u>thm</u> : $f$ conti, $f \geq 0$ on $[a.b]$

$\Rightarrow \lim s(n) = \lim S(n)$

~~***~~ limit indepdent of $f(m_i)$, $f(M_i)$

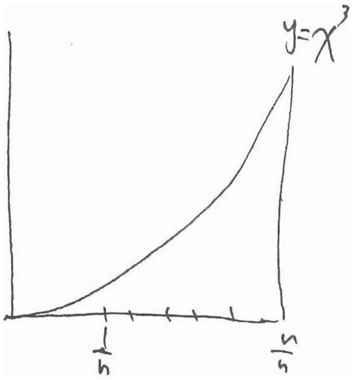
<u>Def</u> : $f \geq 0$. $f$ conti on $[a.b]$

Area of ~~geo~~ region by $f$, x-axis, $x=a$ & $x=b$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

where $x_{i-1} \leq c \leq x_i$

Ex




$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

$$\Delta x = \frac{1}{n}$$

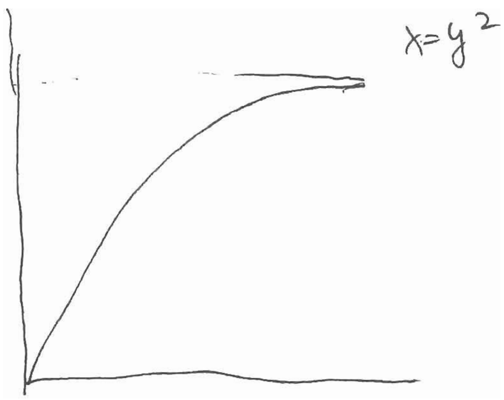
$$c_i = \frac{i}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4}$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \frac{n^2(n+1)^2}{4} = \frac{1}{4}$$

Ex




$$\cancel{\lim}\ \text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta y$$

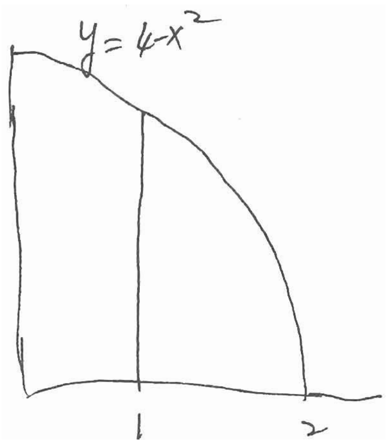
$$\Delta y = \frac{1}{n}$$

$$c_i = \frac{i}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{n^3} \quad \to \frac{1}{3}$$

Ex



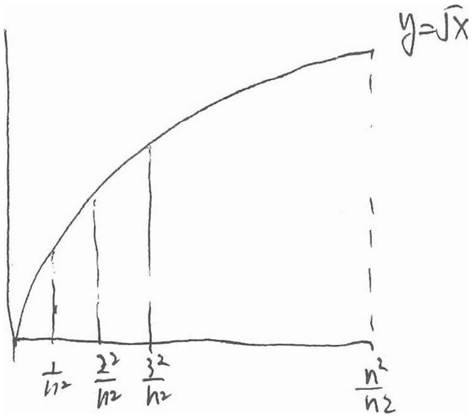

$$c_i = 1 + \frac{i}{n}, \quad \Delta x = \frac{1}{n}$$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(4 - \left(1+\frac{i}{n}\right)^2\right) \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(3 - \frac{2i}{n} - \frac{i^2}{n^2}\right) \frac{1}{n}$$

$$\to \frac{5}{3}$$

$4 - 2 - 3$




$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

$$x_1 = \frac{1}{n^2} \qquad x_i = \frac{i^2}{n^2}$$

$$x_2 = \frac{2^2}{n^2} \qquad \vdots$$

$$x_3 = \frac{3^3}{n^3} \qquad x_n = \frac{n^2}{n^2} = 1$$

$$\Delta x_i = x_i - x_{i-1} = \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2}$$

$$= \frac{2i-1}{n^2}$$

$$c_i = \frac{i^2}{n^2}$$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} \sqrt{\frac{i^2}{n^2}} \cdot \left(\frac{2i-1}{n^2}\right)$$

$$= \sum_{i=1}^{n} \frac{i}{n^2} \cdot \frac{2i-1}{n^2}$$

$$= \frac{1}{n^3}\left(\sum_{i=1}^{n} 2i^2 - \sum_{i=1}^{n} i\right) = \frac{1}{n^3}\left(2\cdot\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right)$$

$$\longrightarrow \frac{1}{3}$$

<u>Def</u>. $\Delta$ partition of $[a,b]$

i.e. $\Delta = \{x_0, x_1, x_2, \cdots x_n\}$ such that

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b, \qquad \Delta x_i = x_i - x_{i-1}$$

Take $c_i$ from $[x_{i-1}, x_i]$ i.e. $x_{i-1} \le c_i \le x_i$

the Sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

is called <u>Riemann Sum</u> of $f$ for the partition $\Delta$

$$\|\Delta\| := \max\{\Delta x_1, \Delta x_2, \cdots, \Delta x_n\}$$

Def : The limit

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx$$

is called the <u>definite integral</u> of $f$ from $a$ to $b$.

$a$    <u>lower limit</u> of integration

$b$    <u>upper limit</u>

$f$    <u>integrable</u> on $[a, b]$

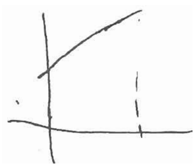
Thm   $f$ conti $\Rightarrow$ $f$ integrable.

~~Rather~~ :
Thm : $f$ conti, $f \geq 0$ on $[a, b]$

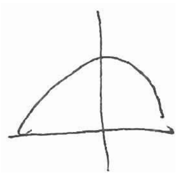
$\Rightarrow$ ~~Area~~ Area of region bounded by $f$, x-axis, $x=a$, $x=b$ $= \int_a^b f(x)\,dx$

Ex   $\int_1^3 4\,dx$

$\int_0^3 x+2\,dx$



$\int_{-2}^{2} \sqrt{4-x^2}\,dx$



Rule:

(i) $\int_a^a f(x)\,dx = 0$

(ii) $\int_b^a f(x)\,dx = -\int_b^a f(x)\,dx$

(iii) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

(iv) $\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$

(v) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(v) $f \geq 0 \Rightarrow 0 \leq \int_a^b f(x)\,dx$

(I) $f(x)$ conti on $[a,b]$

$F'(x) = f(x)$

$$\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

(II) $f(x)$ conti on $[a,b]$

$$\rightarrow \frac{d}{dx}\int_a^x f(x)\,dx = F(x)$$

** Mean Value Theorem for Integrals

$f$ conti on $[a,b]$

$\Rightarrow$ There exists a $c \in [a,b]$, such that

$$\int_a^b f(x)\,dx = f(c)\cdot(b-a)$$

Ex $\displaystyle\int_1^4 3\sqrt{x}\,dx = 3\cdot\frac{1}{\frac{3}{2}}x^{\frac{3}{2}}\Big|_1^4$

Ex $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x\,dx = \tan x\Big|_0^{\frac{\pi}{4}}$

Ex $\displaystyle\int_0^2 |2x-1|\,dx = \int_0^{\frac{1}{2}} -(2x-1)\,dx + \int_{\frac{1}{2}}^2 (2x-1)\,dx$

Ex: $F(x) = \int_0^x \cos t \, dt$

$= \sin t \Big|_0^x$

$= \sin x - \sin 0 = \sin x$

$x=0 \quad F(0) = \int_0^0 \cos t \, dt = \sin 0 = 0$

$x=\frac{\pi}{6} \quad F(\frac{\pi}{6}) = \int_0^{\frac{\pi}{6}} \cos t \, dt = \sin \frac{\pi}{6} = \frac{1}{2}$

$x=\frac{\pi}{4} \quad F(\frac{\pi}{4}) = \int_0^{\frac{\pi}{4}} \cos t \, dt = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Ex $\frac{d}{dx} \int_0^x \sqrt{x^2+1} \, dt$

$= \sqrt{x^2+1}$

Ex $F(x) = \int_{\frac{\pi}{2}}^{x^3} \cos t \, dt$

$F'(x) = \frac{d}{dx} \int_{\frac{\pi}{2}}^{x^3} \cos t \, dt = \frac{du}{dx} \frac{d}{du} \int_{\frac{\pi}{2}}^{u} \cos t \, dt \quad , u = x^3$

$= 3x^2 \cdot \cos u = 3x^2 \cdot \cos x^3$

Ex $F(x) = \int_0^{x^3} \sin \theta^2 \, d\theta$

Ex $F(x) = \int_{x}^{x+2} (4t+1)\, dt$

Ex $f(x) = \int_{0}^{\frac{1}{x}} \frac{1}{t^2+1}\, dt + \int_{0}^{x} \frac{dt}{t^2+1}$

Ex $G(x) = \int_{0}^{x} \left[ s \int_{0}^{s} f(t)\, dt \right] ds$

(a) $G(0)$

(b) $G'(0)$

(c) $G''(x)$

(d) $G''(0)$

# 4-5

$$\frac{d}{dx} F(g(x)) = F'(g(x))\ g'(x)$$

$$\Rightarrow \int F'(g(x))\, g'(x)\, dx = F(g(x)) + C$$

Ex $\int (x^2+1)^2 (2x)\, dx$

$u = x^2+1$

$du = 2x\, dx$

$$= \int u^2 \cdot du = \frac{1}{3} u^3 + C = \frac{1}{3}(x^2+1)^3 + C$$

Ex $\int \sqrt{2x-1}\, dx$

$u = 2x-1 \qquad du = 2dx \Rightarrow dx = \frac{du}{2}$

$$= \int \sqrt{u} \cdot \frac{du}{2}$$

$$= \frac{1}{2} \frac{1}{\frac{3}{2}} u^{\frac{3}{2}} + C$$

$$= \frac{1}{3} (2x+1)^{\frac{3}{2}} + C$$

**Ex** $\displaystyle\int x\sqrt{2x-1}\;dx$

$u = 2x-1 \quad \Rightarrow \quad x = \dfrac{u+1}{2}$

$du = 2\,dx \quad \Rightarrow \quad dx = \dfrac{1}{2}\,du$

$$= \int \left(\frac{u+1}{2}\right) u^{\frac{1}{2}} \cdot \frac{1}{2}\,du$$

$$= \frac{1}{4}\int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$$

$$= \frac{1}{4}\left(\frac{1}{\frac{5}{2}}u^{\frac{5}{2}} + \frac{1}{\frac{3}{2}}u^{\frac{3}{2}}\right) + C$$

$$= \frac{1}{10}(2x-1)^{\frac{5}{2}} + \frac{1}{6}(2x-1)^{\frac{3}{2}} + C.$$

**Ex** $\displaystyle\int_0^1 x\,(x^2+1)^3\,dx$

$u = x^2+1 \qquad du = 2x\,dx$

$x=0\ , \quad u = 0^2+1 = 1$

$x=1\ , \quad u = 1^2+1 = 2$

$$= \int_1^2 u^3\,\frac{du}{2} = \frac{1}{2}\,\frac{u^4}{4}\Bigg|_1^2$$

답

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx$$

$u=\sqrt{2x-1}$ $\quad u^2=2x-1 \Rightarrow 2u\,du = 2dx$

or $\quad x = \dfrac{u^2+1}{2}$

$x=1 \quad u=\sqrt{2\cdot 1-1} = 1$

$x=5 \quad u=\sqrt{2\cdot 5-1} = \sqrt{9} = 3$

$$= \int_1^3 \frac{1}{u}\cdot\frac{u^2+1}{2}\cdot u\,du = \frac{1}{2}\int_1^3 (u^2+1)\,du$$

$$= \frac{1}{2}\left(\frac{1}{3}u^3+u\right)\Big|_1^3$$

$$= \frac{1}{2}\left(\frac{1}{3}\cdot 3^3+3\right) - \frac{1}{2}\left(\frac{1}{3}\cdot 1^3+1\right) = \frac{16}{3}$$