Def f defined on I, $c \in I$.

(i) $f(c)$ (absolute) <u>minimum</u> of f on I if $f(c) \le f(x)$ $\forall x \in I$

(ii) $f(c)$ <u>maximum</u> of f on I if $f(c) \ge f(x)$ $\forall x \in I$

(iii) <u>extreme value</u> or <u>extrema</u> of f on I = minimum or maximum.

Thm f <u>continuous</u> on $[a, b]$

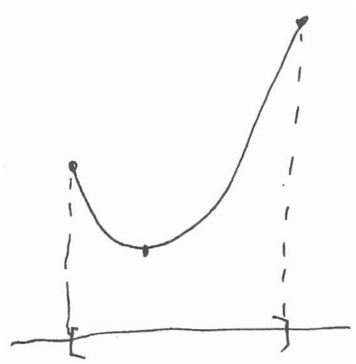
$\Rightarrow$ f has a minimum and a maximum on $[a, b]$

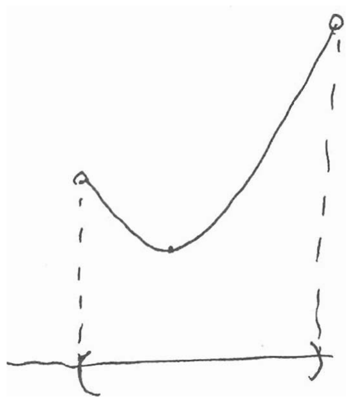
Def (i) $f(c)$ <u>relative maximum</u> if there is (a, b), $c \in (a, b)$

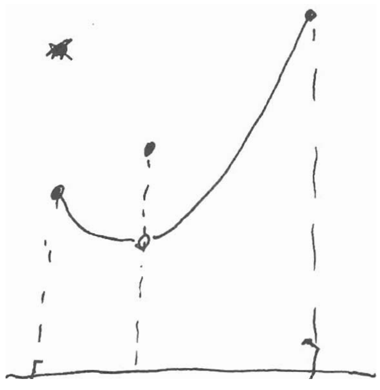
$f(c)$ maximum on (a, b)

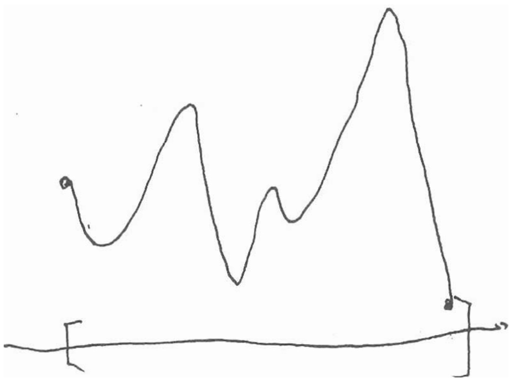
(ii) $f(c)$ <u>relative minimum</u> if there is (a, b), $c \in (a, b)$

$f(c)$ minimum on (a, b)









Def : critical number c of f if $f'(c)=0$ or

f not differentiable at c.

Thm : f has a relative minimum or a relative maximum at $x=c$

$\Rightarrow$ c critical number of f

Example : $f(x)=3x^4-4x^3$ on $[-1,2]$

$f'(x) = 12x^3-12x^2=0$ $\quad x=0, 1$

-1	0	1	2
7	0	-1	16

Ex $f(x) = 2x - 3X^{3}$ on $[-1, 3)$

$f'(x) = 2 - 2 \cdot X^{-\frac{1}{3}} = 2\left(1 - \frac{1}{X^{\frac{1}{3}}}\right)$.

$\left.\begin{array}{ll} X=1 & f'(1) = 0 \\ X=0 & f'(x) \text{ not exist} \end{array}\right\}$ critical point

-1	0	1	3
-5	0	-1	$6-3\sqrt[3]{9} \approx -0.24$

Ex $f(x) = 2\sin x - \cos 2x$ on $[0, 2\pi]$

$f'(x) = 2\cos x + 2\sin 2x$

$= 2\cos x + 2 \cdot 2\sin x \cos x$

$= 2\cos x\,(1 + 2\sin x) = 0$

$X = \frac{\pi}{2}, \frac{3\pi}{2}$ $\cos X = 0$

$X = \frac{7\pi}{6}, \frac{11\pi}{6}$ $\sin X = -\frac{1}{2}$

0	$\frac{\pi}{2}$	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
-1	3	$-\frac{3}{2}$	-1	$-\frac{3}{2}$	-1

3-2

Thm (Rolle's Thm)

f contin on $[a,b]$

f diff on (a,b)

$f(a) = f(b)$

$\Rightarrow \exists\, c \in (a,b)$, $f'(c) = 0$

pf (I) Case I : $f(a) = f(b) = f(x) =$ const $\forall x$

(II) Case II : $f(x)$ not constant.

(i) $f(x) > f(a) = f(b)$ max

(ii) $f(x) < f(a) = f(b)$ min $\Big\} \rightarrow$ critical point $\rightarrow f'(c) = 0$

Ex $f(x) = x^2 - 3x + 2$

$\Rightarrow f'(x) = 0$ at some point between two x-intercepts.

Ex $f(x) = x^4 - 2x^2$ on $(-2, 2)$

$f'(x) = 4x^3 - 4x = 0$

$\Rightarrow x = 0, 1, -1$

Thm: Mean Value Thm

f contin on $[a, b]$

f diff on (a, b)

$\Rightarrow \exists\, c \in (a, b), \quad f'(c) = \dfrac{f(b) - f(a)}{b - a}$

pf $\quad y = \left[\dfrac{f(b) - f(a)}{b - a}\right](x - a) + f(a)$

Rem : $f(b) = f(a) + f'(c)(b - a)$

Def : f increasing if for any $x_1 < x_2$, $f(x_1) < f(x_2)$

f decreasing if for any $x_1 < x_2$, $f(x_1) > f(x_2)$

Thm : f conti on $[a,b]$, diff on (a,b)

(i) $f'(x) > 0 \quad \forall x \in (a,b) \rightarrow f$ increasing on $[a,b]$

(ii) $f'(x) < 0 \quad \forall x \in (a,b) \rightarrow f$ decreasing on $[a,b]$

(iii) $f'(x) = 0 \quad \forall x \in (a,b) \rightarrow f$ constant on (a,b)

pf (i) ~~$x_1 < c < x_2$~~ By M.V.T $\exists c$, $x_1 < c < x_2$

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \quad \Rightarrow \quad f(x_2) > f(x_1)$$

(ii) $f'(c) < 0$, $f(x_2) < f(x_1)$

(iii) $f'(c) = 0$ $\quad f(x_2) = f(x_1)$

Ex 1 $f(x) = x^3 - \frac{3}{2}x^2$

$f'(x) = 3x^2 - 3x$

$-\infty < x < 0$	$0 < x < 1$	$1 < x < \infty$

Thm (First Derivative Test) c critical number of f

(i) $f'(x) < 0 \longrightarrow 0 < f'(x)$, f relative min.

(ii) $0 < f'(x) \longrightarrow f'(x) < 0$, f relative max

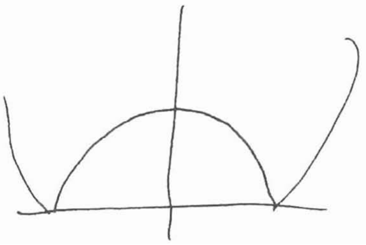
(iii) $\left.\begin{array}{l} f'(x) > 0 \\ \text{or} \\ f'(x) < 0 \end{array}\right\}$ on both side of c $\longrightarrow$ $f(c)$ neither rel. max nor rel min

Ex $f(x) = \frac{1}{2}x - \sin x$ on $(0, 2\pi)$

$f'(x) = \frac{1}{2} - \cos x$

	$0 < x < \frac{\pi}{3}$	$\frac{\pi}{3} < x < \frac{5\pi}{3}$	$\frac{5\pi}{3} < x < 2\pi$
Test value	$\frac{\pi}{4}$	π	$\frac{7\pi}{4}$
	$f'(\frac{\pi}{4}) < 0$	$f'(\pi) > 0$	$f'(\frac{7\pi}{4}) < 0$

Ex $f(x) = (x^2-4)^{\frac{2}{3}}$



$f'(x) = \dfrac{4x}{3(x^2-4)^{1/3}}$

$x = \pm 2$ $f'(0)$ not exist $\longrightarrow$ $x = \pm 2, 0$ critical point

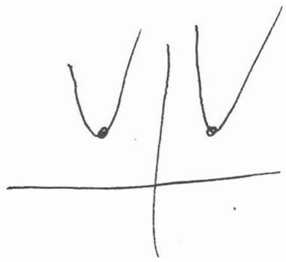
$x=0$, $f'(0) = 0$

$x<-2$	$-2<x<0$	$0<x<2$	$2<x$
-3	-1	1	3
$f'(-3)<0$	$f'(-1)>0$	$f'(1)<0$	$f'(3)>0$

Ex $f(x) = \dfrac{x^4+1}{x^2}$

$f'(x) = \dfrac{2(x^2+1)(x-1)(x+1)}{x^3}$. critical number ± 1

$-\infty<x<-1$	$-1<x<0$	$0<x<1$	$1<x$
-2	$-\frac{1}{2}$	$\frac{1}{2}$	2.

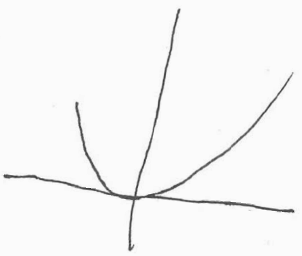


3-4

Def : f differentiable on $(a.b)$

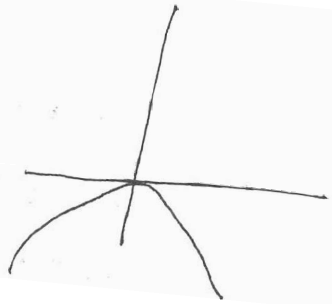
f concave upward on $(a.b)$ if f' increasing on $(a.b)$

concave downward on $(a.b)$ if f' decreasing on $(a.b)$



$f(x) = x^2$

$f'(x) = 2x$



$f(x) = -x^2$

$f'(x) = -2x$

Thm : f'' exists on $(a.b)$

$\Rightarrow$ (i) $f''(x) > 0 \quad \forall x \in (a.b) \quad \Rightarrow f(x)$ concave upward

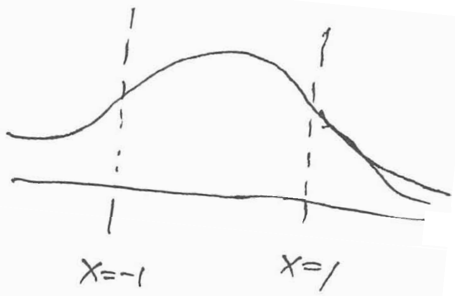
(ii) $f''(x) < 0 \quad \forall x \in (a.b) \quad \Rightarrow f(x)$ concave downward

Ex $\quad f(x) = \dfrac{6}{x^2+3} \qquad f'(x) = \dfrac{-12x}{(x^2+3)^2}$

$$f''(x) = \frac{36(x^2-1)}{(x^2+3)^3}$$

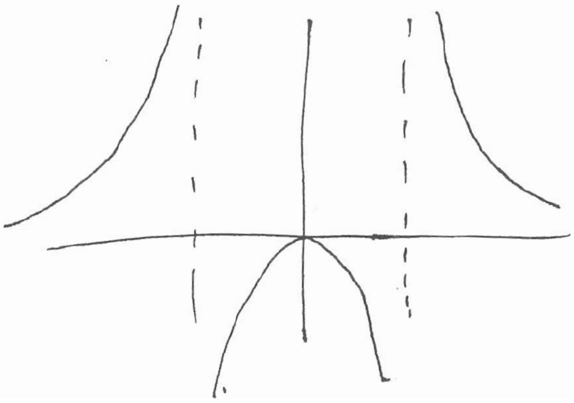
$f''(x) = 0 \quad \Rightarrow \quad x = \pm 1$

$x < -1$	$-1 < x < 1$	$1 < x$
$f''(-2) > 0$	$f''(0) = 0$	$f''(2) > 0$



Ex $f(x) = \dfrac{x^2+1}{x^2-4} = 1 + \dfrac{5}{x^2-4}$

$f'(x) = \dfrac{-10x}{(x^2-4)^2}$, $f''(x) = \dfrac{10(3x^2+4)}{(x^2-4)^3}$



Def: $(c, f(c))$ point of inflection if ~~the~~ at $(c, f(c))$ the concavity of f changes

~~Def~~ Thm: $(c, f(c))$ point of inflection

$\Longrightarrow$ either $f''(c) = 0$ or f'' not exist at $x = c$.

Ex $f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x$

$\boxed{\text{Thm}}$ $f'(c) = 0$

(i) $f''(c) > 0$, $\Rightarrow f(c)$ relative minimum

(ii) $f''(c) < 0$ $\Rightarrow f(c)$ relative maximum

(iii) $f''(c) = 0$ $\not\Rightarrow$ ~~no~~ test fails

Ex

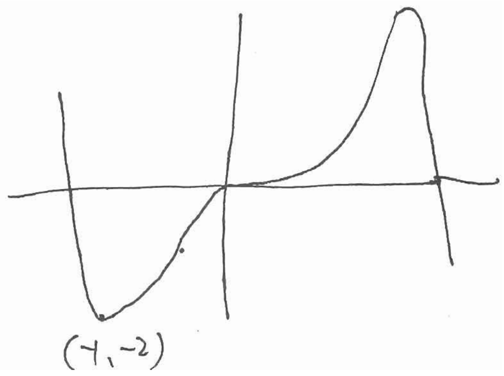
$f(x) = -3x^5 + 5x^3$

$f'(x) = -15x^4 + 15x^2 \Rightarrow x = -1, 0, 1$

$f''(x) = -60x^3 + 30x$

① $f''(-1) = (-60)(-1)^3 + 30(-1) = 60 - 30 = 30 > 0$

$f(-1) = (-3)(-)^5 + 5(-)^3 = -2$ relative min



② $f''(1) = -60 + 30 = -30 < 0$

$f(1) = -3 + 5 = 2$ relative max

③ $f''(0) = 0$,

$f(0) = 0$

$f'(-\frac{1}{2}) = -\frac{15}{16} + 15\,\frac{1}{4} > 0$

$f'(\frac{1}{2}) = -\frac{15}{16} + \frac{15}{4} > 0$

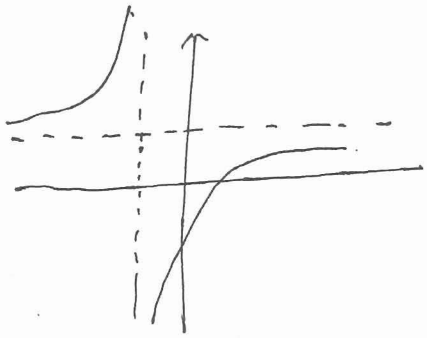
$\Rightarrow f(0)$ neither max nor min

3-5 $\lim_{x\to\infty}$, $\lim_{x\to-\infty}$

Def $y=L$ <u>horizontal asymptote</u> of f

if $\lim_{x\to\infty} f(x)=L$ or $\lim_{x\to-\infty} f(x)=L$

Ex $\lim_{x\to\infty} \frac{2x-1}{x+1} = \lim_{x\to\infty} \frac{2-\frac{1}{x}}{1+\frac{1}{x}} = 2$



Ex $\lim_{x\to\infty} \frac{2x+5}{3x^2+1}$, $\lim_{x\to\infty} \frac{2x^2+5}{3x^2+1}$, $\lim_{x\to\infty} \frac{2x^3+5}{3x^2+1}$

Ex $\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}}$

Ex $\lim_{x\to\infty} \sin x$

Ex $\lim_{x\to\infty} \frac{\sin x}{x}$

Ex $\lim_{x \to \infty} x^3$

$\lim_{x \to -\infty} x^3$

Ex $\lim_{x \to \infty} \frac{2x^2-4x}{x+1}$

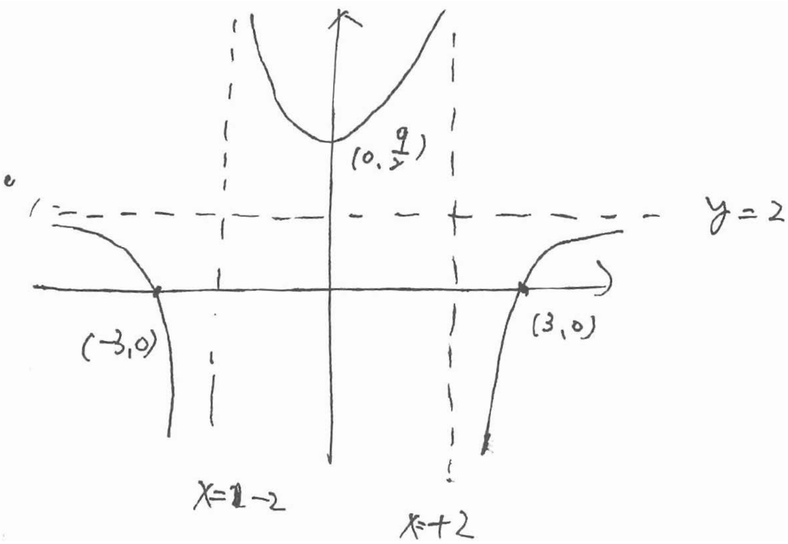
$\lim_{x \to -\infty} \frac{2x^2-4x}{x+1}$

3-6

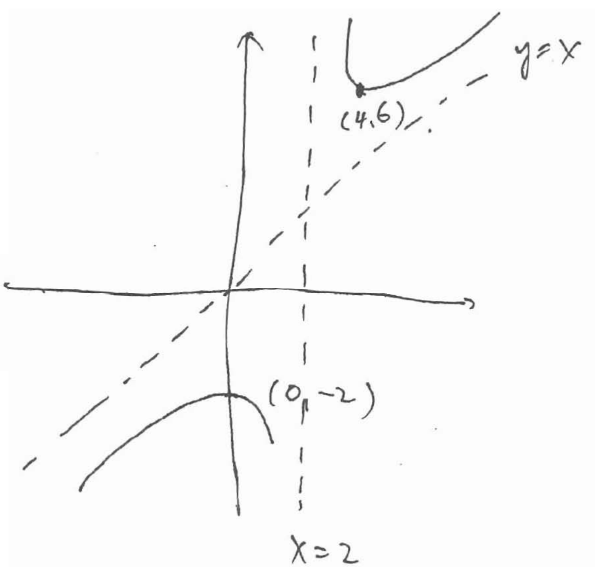
① $f(x) = \dfrac{2(x^2-9)}{x^2-4} = 2 - \dfrac{10}{x^2-4}$

$f'(x) = \dfrac{20x}{(x^2-2)^2}$

$f''(x) = \dfrac{-20(3x^2+4)}{(x^2-2)^3}$ no inflection point



② $f(x) = \dfrac{x^2-2x+4}{x-2} = x + \dfrac{4}{x-2}$



slant asymptote

$f'(x) = \dfrac{-4}{(x-2)^2} + 1 = \dfrac{x(x-x)}{(x-2)^2}$

$f''(x) = \dfrac{8}{(x-2)^3}$

Ex $f(x) = 2x^{\frac{5}{3}} - 5x^{\frac{4}{3}}$

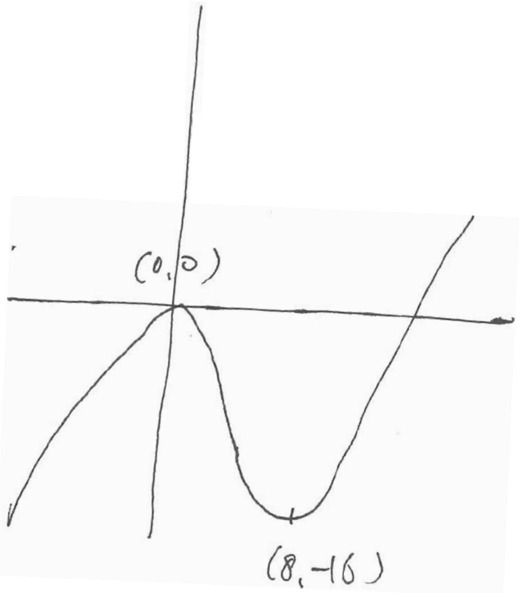
$f'(x) = \frac{10}{3}x^{\frac{2}{3}} - \frac{20}{3}x^{\frac{1}{3}} = \frac{10}{3}x^{\frac{1}{3}}(x^{\frac{1}{3}} - 2)$

$= 0$

$\Rightarrow x = 0, \ 8$

$f''(x) = \frac{20}{9}x^{-\frac{1}{3}} - \frac{20}{9}x^{-\frac{2}{3}}$

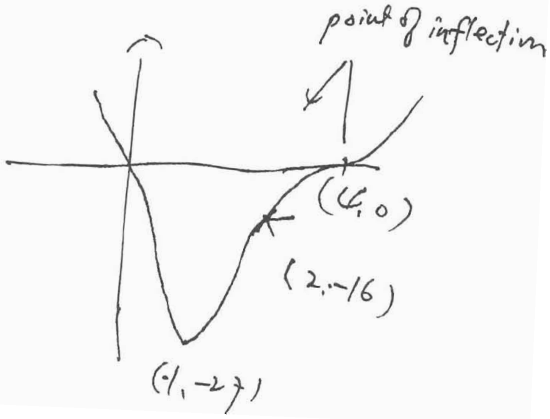
possible point of inflection : $x = 0, \ 1$



Ex $f(x) = x^4 - 12x^3 - 48^2 - 64x = x(x-4)^3$

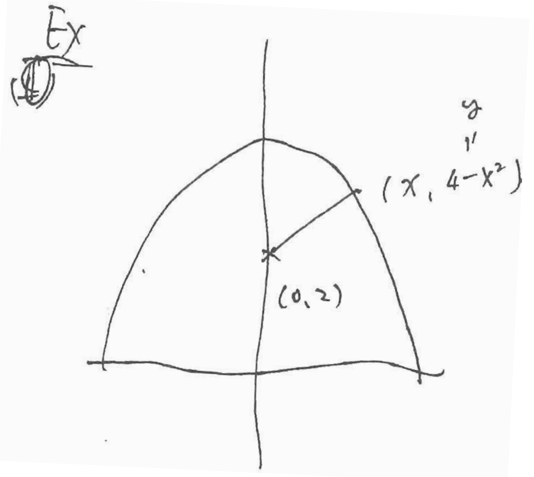
$f'(x) = 4(x-1)(x-4)^2$

$f''(x) = 12(x-4)(x-2)$



3-7

Ex (1)



$d = \sqrt{(x-0)^2 + (y-2)^2}$

$= \sqrt{x^2 + (4-x^2-2)^2}$

$= \sqrt{x^4 - 3x^2 + 4}$

$f(x) = x^4 - 3x^2 + 4$

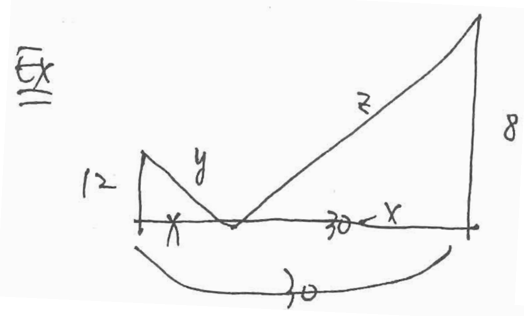
$f'(x) = 4x^3 - 6x = 0$ $4x(x^2 - \frac{3}{2})$ $x = 0$ or $\pm\sqrt{\frac{3}{2}}$

(i) $f''(x) = 12x^2 - 6$ $f''(0) < 0$, $f''(\pm\sqrt{\frac{3}{2}}) > 0$

(ii) $f'(1) < 0$, $f'(-1) > 0$

$f'(2) > 0$ $f'(-2) < 0$

Ex



$W = y + z$

$x^2 + 12^2 = y^2$

$(30-x)^2 + 28^2 = z^2$

$W = \sqrt{x^2+144} + \sqrt{x^2 - 60x + 1684}$

$\frac{dW}{dx} = \frac{x}{\sqrt{x^2+144}} + \frac{x-30}{\sqrt{x^2-60x+1684}} = 0$

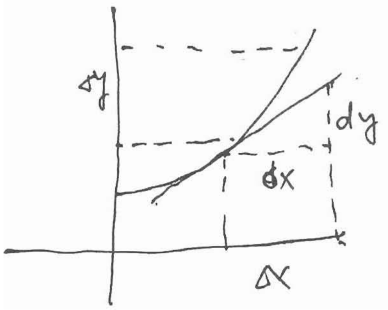
$x\sqrt{x^2-60x+1684} = -(x-30)\sqrt{x^2+144}$

$\Rightarrow 640x^2 + 8640x - 129{,}600 = 0 \Rightarrow x = 9, -22.5$

3-9

$\Delta x = dx$, dx : differential of x

$dy = f'(x)\,dx$: differential of y



$\Delta y = f(x+\Delta x) - f(x)$

** In many application ~~...~~ Δy is approximately equal to dy i.e.,

$$\Delta y \approx dy = f'(x)\,dx$$

Ex $\sqrt{16.5} = ?$

$f(x) = \sqrt{x}$ with $x = 16$; $\Delta x = 0.5$ $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$$\Delta y = f(16+0.5) - f(16)$$

$$\approx dy$$

$$= f'(x)\cdot dx$$

$$= \frac{1}{2}(16)^{-\frac{1}{2}}\cdot(0.5) = \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{2} = 0.0625$$

$$f(16.5) \approx f(16) + dy = \sqrt{16} + 0.0625$$

$$= 4.0625$$

Rule : $d[cu] = c\,du$

$d[u \pm v] = du \pm dv$

$d[uv] = v\,du + u\,dv$

$d\left[\frac{u}{v}\right] = \frac{v\,du - u\,dv}{v^2}$.

Ex $\tan 0.05 = ?$

$f(x) = \tan x$, $x = 0$, $\Delta x = dx = 0.05$

$f'(x) = \sec^2 x$

$\Delta y = \tan 0.05 - \tan 0$

$= f'(x) \cdot dx$

$= \sec^2 x \cdot dx$

$= \sec^2 0 \cdot (0.05)$

$= 1 \cdot (0.05)$

$\Rightarrow \tan 0.05 = \tan 0 + 0.05 = 0.05$