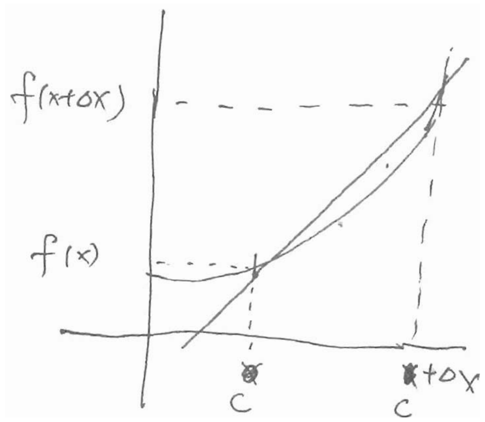
/ 2.1



Δx change in x

$\Delta y = f(x+\Delta x) - f(x)$ change in y

Def : ~~slope of the tan~~ The f defined on (a, b). $c \in (a, b)$

the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m$$

is called the slope of the tangent line to the graph of f at $(c, f(c))$

Ex Find the slope of the tangent line to the graph. $f(x) = x^2+1$ at $(0, 1)$ and $(-1, 2)$

Sol : The slope of the tangent line at $(c, f(c))$ is given by

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left((c+\Delta x)^2+1\right) - (c^2+1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{c^2 + 2c\cdot\Delta x + (\Delta x)^2 - c^2}{\Delta x}$$

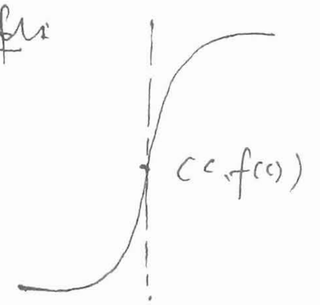
$$= \lim_{\Delta x \to 0} 2c + \Delta x$$

$$= 2c.$$

$c = 0$ $m = 0$

$c = -1$ $m = -2$

Def.



vertical tangent line

Def If f contin at c, and

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \infty$$

or $$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = -\infty,$$

$x = c$: vertical tangent line

Def : The derivative of f at x is given by

$$f'(x) := \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If $f'(x)$ exists at x, $f(x)$ is called differentiable at x.

If $f'(x)$ exist at every x, $x \in (a,b)$, $f(x)$ is called differentiable on (a,b).

Notation : $f'(x)$, $\frac{dy}{dx}$, y', $\frac{d}{dx}x$, $D_x f$

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Ex $f(x) = x^3 + 2x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = 3x^2 + 2$$

Ex $f(x) = \sqrt{x}$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Def $\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$ derivative from the left

$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$ derivative from the right

Ex : $f(x) = |x-2|$

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2} = -1$$

$$\lim_{x \to 2^+} \frac{f(x) - f(2)}{x-2} = 1$$

$\Rightarrow$ f not differentiable at $x=2$.

Ex $f(x) = x^{\frac{1}{3}}$ continuous at $x=0$.

$$\lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{x - 0} = \lim_{\Delta x \to 0} \frac{(\Delta x)^{\frac{1}{3}}}{\Delta x} = \lim_{\Delta x \to 0} (\Delta x)^{-\frac{2}{3}} = \infty$$

$f(x)$ not differentiable at $x=0$

Thm : f(x) differentiable at $x=c$

$\Rightarrow$ f(x) continuous at $x=c$

pf

$$\lim_{x\to c}(f(x)-f(c)) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot(x-c)$$

$$= \lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot\lim_{x\to c}(x-c)$$

$$= f'(c)\cdot 0$$

$$= 0$$

2-2 Basic Rules

thm 2.2 $\frac{d}{dx}c = 0$

Thm 2.3 $\frac{d}{dx}x^n = nx^{n-1}$ (in general $n \in \mathbb{R}$)

To $\frac{d}{dx}x^n = \lim_{\Delta x\to 0}\frac{(x+\Delta x)^n - x^n}{\Delta x}$

thm 2.4 $\frac{d}{dx}c\cdot f(x) = c\cdot\frac{d}{dx}f(x)$

thm 2.5 $\frac{d}{dx}(f(x)\pm g(x)) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

$$y' = \frac{d}{dx}\frac{2}{x} = 2\frac{d}{dx}x^{-1} = 2\cdot(-1)x^{-2}$$

Ex $f(t) = 3\sqrt{t}$

$$f'(t) = 3(\sqrt{t})' = 3(t^{\frac{1}{2}})' = 3\cdot\frac{1}{2}t^{-\frac{1}{2}}$$

Ex $y = \dfrac{1}{2\sqrt[3]{x^2}}$

$$y' = \frac{1}{2}\frac{d}{dx}x^{-\frac{2}{3}} = \frac{1}{2}\left(-\frac{2}{3}\right)x^{-\frac{5}{3}} = -\frac{1}{3x^{\frac{5}{3}}}$$

Ex $f(x) = x^3 - 4x + 5$

$$f'(x) = (x^3)' - 4(x)' + 5'$$

$$= 3x^2 - 4x$$

Thm : $\dfrac{d}{dx}\sin x = \cos x$, $\dfrac{d}{dx}\cos x = -\sin x$

pf $\dfrac{d}{dx}\sin x = \lim\limits_{\Delta x \to 0}\dfrac{\sin(x+\Delta x) - \sin x}{\Delta x}$

$$= \lim_{\Delta x \to 0}\frac{\sin x\cos\Delta x + \cos x\sin\Delta x - \sin x}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\sin x\left(\frac{\cos\Delta x - 1}{\Delta x}\right) + \lim_{\Delta x \to 0}\cos x\,\frac{\sin\Delta x}{\Delta x}$$

$$= \cos x$$

2.3

Thm $\frac{d}{dx}[f(x)g(x)] = f'(x)\cdot g(x) + f(x)\cdot g'(x)$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g^2(x)}$$

Ex $y = 3x^2 \sin x$

$y' = (3x^2)' \cdot \sin x + 3x^2 \cdot (\sin x)'$

$= 6x \sin x + 3x^2 \cos x$

Ex $y = \frac{5x-2}{x^2+1}$

$$y' = \frac{(5x-2)'(x^2+1) - (5x-2)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{5(x^2+1) - (5x-2)\cdot 2x}{(x^2+1)^2} = \frac{-5x^2+4x+5}{(x^2+1)^2}$$

Thm

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{(\sin x)'\cos x - \sin x\,(\cos x)'}{\cos^2 x}$$

$$= \frac{\cos^2 x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \sec^2 x$$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{d}{dx}\frac{(1)'\cos x - 1(\cos x)'}{\cos^2 x} = \frac{-(-\sin x)}{\cos^2 x}$$

$$= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

pf $\quad \frac{d}{dx}\left[f(x)\cdot g(x)\right] = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)\,g(x+\Delta x) - f(x)\,g(x)}{\Delta x}$

Ex $\quad y = \frac{1-\cos x}{\sin x}$

$$y' = \frac{(1-\cos x)'\sin x - (1-\cos x)\sin x}{\sin^2 x}$$

or

$$y = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \csc x - \cot x$$

$$y' = \csc^2 x - \csc x \cot x$$

First derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$D_x f$
	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$D_x^2 f$
	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$D_x^3 f$
	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$D_x^4 f$
	$\vdots$	$\vdots$		
	$y^{(n)}$	$f^{(n)}(x)$		$D_x^n f$

Ex $\quad s(t) = -0.81t^2+2$

$s'(t) = -1.62t$

$s''(t) = -1.62$

2-4 Chain Rule

~~Thm~~: $x \longmapsto g(x) = u \longmapsto y = f(u) = f(g(x))$

Ex $\quad x \longmapsto x^2 = u \longmapsto y = \sin u = \sin x^2$

Thm: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

Ex $\quad \dfrac{d}{dx} \sin x^2 = \dfrac{d}{du} \sin u \cdot \dfrac{du}{dx} \qquad u = x^2$

$$= \cos u \cdot 2x$$

$$= \cos x^2 \cdot 2x$$

Ex $\quad x \longmapsto u = \sin x \longmapsto y = u^2 = \sin^2 u$

$$\frac{d \sin^2 x}{dx} = \frac{d}{du} u^2 \cdot \frac{du}{dx}$$

$$= 2u \cdot \cos x$$

$$= 2 \sin x \cdot \cos x$$

Proof: $\dfrac{dy}{dx} = \lim\limits_{x \to c} \dfrac{f(g(x)) - f(g(c))}{x - c}$

$$= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c}$$

$$= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

$$= \lim_{u \to g(c)} \frac{f(u) - f(g(c))}{u - g(c)} \cdot \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

$$= \frac{df(u)}{du} \cdot \frac{du}{dx}$$

Ex $f(x) = (3x - 2x^2)^3$

Ex $f(x) = \sqrt[3]{(x^2-1)^2}$

$\begin{cases} f'(x) = 0. \\ f'(x) \text{ not exist} \end{cases}$

Ex $f(x) = x^2\sqrt{1-x^2}$

Ex $y = \tan(x-1)$

Ex $y = \sqrt{\cos x}$

$\cancel{h(x)\ f(g)\ y = f(g(h(x)))}$

$\cancel{\frac{dy}{dx}} = \frac{df(u)}{dg} \cdot \frac{dg(x)}{d}$

Ex $f(t) = \sin^3 4t$

** Summary

2.5

$x^2+y^2=25$

$y_1 = +\sqrt{25-x^2}$ or $y_2 = -\sqrt{25-x^2}$

$\frac{dy_1}{dx} = \frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x)$ or $\frac{dy_2}{dx} = -\frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x)$

$= \frac{-x}{(25-x^2)^{1/2}} = \frac{-x}{y}$

$= \frac{x}{+(25-x^2)^{1/2}}$

$= \frac{x}{-y}$

More complicated ?

$\Rightarrow$ implicit differentiation

$x^2+y^2 = 25$

$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25) = 0$

$2x + 2y\frac{dy}{dx} = 0 \Rightarrow 2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

Ex $x^2-2y^3+4y = 2$

$\frac{d}{dx}(x^2-2y^3+4y) = \frac{d}{dx}2 = 0$

$2x - 2\cdot 3y^2\frac{dy}{dx} + 4\frac{dy}{dx} = 0$

$(-6y^2+4)\frac{dy}{dx} = 2x$

$\frac{dy}{dx} = \frac{2x}{-6y^2+4}$

Ex

$$y^3 + y^2 - 5y - x^2 = -x$$

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$

Ex

$$x^2 + 4y^2 = 4 \quad \text{at } \left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right)$$

$$\frac{dy}{dx} = \frac{-x}{4y} = \frac{1}{2}$$

Ex $\sin y = x$ $\qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

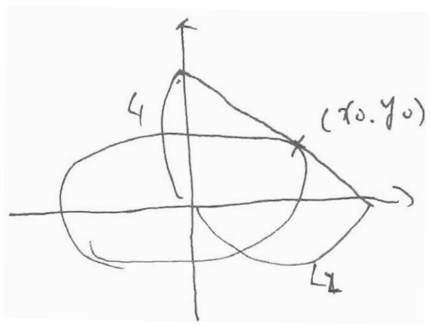
Ex

$$x^2 + y^2 = 25$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{x^2+y^2}{y^3} = -\frac{25}{y^3}$$

Ex 74



$$\sqrt{x} + \sqrt{y} = \sqrt{c}$$

$$\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}} = -\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

$$y - y_0 = \frac{-y_0^{\frac{1}{2}}}{x_0^{\frac{1}{2}}}(x - x_0)$$

$x_0 = 0$

$y_0 = 0$

$$L_1 = y_0 + \frac{-y_0^{\frac{1}{2}}}{x_0^{\frac{1}{2}}}(-x_0) = y_0 + x_0^{\frac{1}{2}}y_0^{\frac{1}{2}}$$

$$L_2 = x_0^{\frac{1}{2}}y_0^{\frac{1}{2}} + x_0$$

$$L_1 + L_2 = x_0 + y_0 + 2x_0^{\frac{1}{2}}y_0^{\frac{1}{2}} = (\sqrt{x_0} + \sqrt{y_0})^2 = C.$$

2-6

$$V = \frac{\pi}{3} r^2 h$$

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{\pi}{3} r^2 h\right)$$

$$= \frac{\pi}{3}\left(\frac{d}{dt} r^2\right) h + \frac{\pi}{3} r^2 \frac{dh}{dt}$$

$$= \frac{\pi}{3}\left[2r \cdot \frac{dr}{dt} \cdot h + r^2 \cdot \frac{dh}{dt}\right]$$

$\Rightarrow \frac{dv}{dt}$ is ~~a function~~ related to of $\frac{dr}{dt}$ and $\frac{dh}{dt}$

Ex $A = \pi r^2$ $\quad \frac{dr}{dt} = 1$, $r = 4$ $\quad \frac{dA}{dt} = ?$

Sol: $\frac{d}{dt} A = \pi \frac{d}{dt} r^2$

$$= \pi\, 2r \frac{dr}{dt}$$

$$\left.\frac{dA}{dt}\right|_{r=4} = \pi \cdot 2 \cdot 4 \cdot 1 = 8\pi.$$

Ex $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = \frac{9}{2}$.

Find $\frac{dr}{dt}$, if $r = 2$

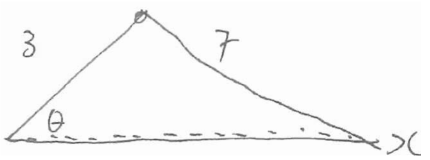
Sol:

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} \quad \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

$$\cancel{\frac{dV}{dt} = \frac{9}{2} = \frac{4}{3}\pi \cdot 3 \cdot 2^2 \frac{dr}{dt}}$$

$$\left.\frac{dr}{dt}\right|_{r=2} = \frac{1}{4\pi \cdot 2^2} \cdot \frac{9}{2} = \frac{9}{32\pi}$$

Ex6



$$\frac{d\theta}{dt} = 200 \cdot 2\pi \ /\text{minute}$$

$$\frac{dx}{dt} = ? \quad , \theta = \frac{\pi}{3}$$

Sol : $\cancel{7^2}$ Law of ~~Con~~ Cosine

$$7^2 = 3^2 + x^2 - 2 \cdot 3 \cdot x \cdot \cos\theta$$

$$\frac{d}{dt}\left(7^2\right) = \frac{d}{dt}\left(3^2 + x^2 - 6x\cos\theta\right) = 0$$

$$2x\frac{dx}{dt} - 6\frac{dx}{dt} \cdot \cos\theta - 6x(-\sin\theta)\frac{d\theta}{dt} = 0$$

$$6x\sin\theta \frac{d\theta}{dt} = (6\cos\theta - 2x)\frac{dx}{dt}$$

$\theta = \frac{\pi}{3}$ $\qquad 7^2 = 3^2 + x^2 - 6x \cdot \underbrace{\cos\frac{\pi}{3}}_{=\frac{1}{2}}$

$$= 3^2 + x^2 - 3x$$

$$x^2 - 3x - 40 = 0 = (x-8)(x+5) \Rightarrow x = 8$$

$$6 \cdot 8 \cdot \sin\frac{\pi}{3} \cdot 400\pi = \left(6 \cdot \cos\frac{\pi}{3} - 2 \cdot 8\right)\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{9600\pi\sqrt{3}}{-13}$$