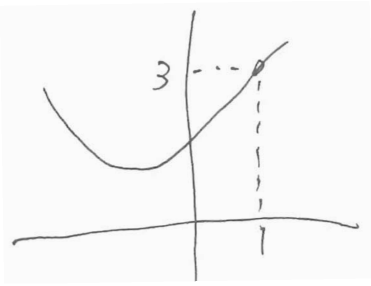
1-2

① $f(x) = \dfrac{x^3-1}{x-1} \quad x \neq 1$

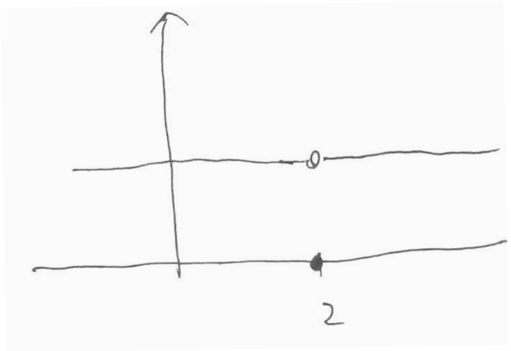


$$\lim_{x \to 1} \frac{x^3-1}{x-1} = 3$$

②

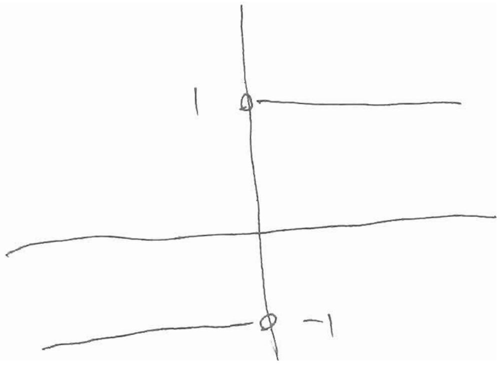
$$\lim_{x \to 0} \frac{x}{\sqrt{x+1}-1} = 2$$

③ $f(x) = \begin{cases} 1 & x \neq 2 \\ 0 & x = 2 \end{cases}$

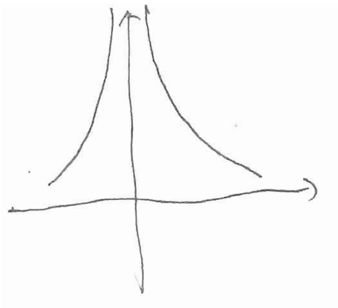


$$\lim_{x \to 2} f(x) = 1$$

④ $\lim \frac{|x|}{x}$ not exist



⑤ $\lim\limits_{x \to 0} \frac{1}{x^2}$



⑥ $\lim\limits_{x \to 1} \sin \frac{1}{x}$ not exist

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$
$\sin\frac{1}{x}$	1	-1	1	-1

1.3

Thm 1.1 ① $\lim_{x \to c} b = b$

② $\lim_{x \to c} x = c.$

③ $\lim_{x \to c} x^n = c^n$

Thm 1.2 Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$

1. $\lim_{x \to c} (b\, f(x)) = b \cdot L$
2. $\lim_{x \to c} (f(x) \pm g(x)) = L \pm M$
3. $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$
4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}$, if $K \neq 0$

Thm 1.3 : $p(x) = a_0 + a_1 x + \cdots + a_n x^n$

, $q(x) = b_0 + b_1 x + \cdots + b_m x^m$, $q(c) \neq 0$

$\Rightarrow \lim_{x \to c} p(x) = a_0 + a_1 c + a_2 c^2 + \cdots + a_n c^n = p(c)$

$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$

Thm 1.4 : $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$, $c > 0$

Thm 1.5 : $\lim_{x \to c} g(x) = L$, $\lim_{x \to L} f(x) = f(L)$

$\Rightarrow \lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L)$

Ex $\lim_{x\to 0}\sqrt{x^2+4} = \sqrt{\lim_{x\to 0}(x^2+4)} = \sqrt{4}$

Ex $\lim_{x\to 6}\sqrt[3]{2x^2-10} = \sqrt[3]{\lim_{x\to 6}(2x^2-10)} = \sqrt[3]{8} = 2$

Thm 1.6

$\lim_{x\to c}\sin x = \sin c$ $\lim_{x\to c}\cos x = \cos c$ $\lim_{x\to c}\tan x = \tan c$

Ex $\lim_{x\to\pi}(x\cos x) = (\lim_{x\to\pi} x)(\lim_{x\to\pi}\cos x) = \pi\cdot\cos\pi = -\pi$

Ex $\lim_{x\to 0}\sin^2 x = (\lim_{x\to 0}\sin x)^2 = (\sin 0)^2 = 0$

Thm 1.7 $f(x) = g(x)$, $x\neq c$. $\lim_{x\to c} g(x) = L$

$\Rightarrow \lim_{x\to c} f(x) = L$

ex1 $\lim_{x\to 1}\frac{x^3-1}{x-1}$

since $\frac{x^3-1}{x-1} = x^2+x+1$ for $x\neq 1$

$\lim_{x\to 1}\frac{x^3-1}{x-1} = \lim_{x\to 1} x^2+x+1 = 3$

ex2 $\lim_{x\to -3}\frac{x^2+x-6}{x+3}$

$x\neq -3$ $\frac{x^2+x-6}{x+3} = \frac{(x+3)(x-2)}{x+3} = x-2$

$\lim_{x\to -3}\frac{x^2+x-6}{x+3} = \lim_{x\to -3} x-2 = -5$

Ex $\displaystyle \lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

$$x \neq 0 \quad \frac{\sqrt{x+1}-1}{x} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \frac{(\sqrt{x+1})^2 - 1^2}{x(\sqrt{x+1}+1)} = \frac{1}{\sqrt{x+1}+1}$$

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+1}+1} = \frac{1}{2}$$

Thm 1.8 ① $h(x) \le f(x) \le g(x)$

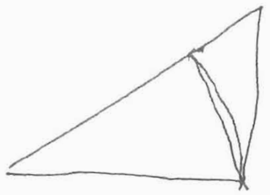
② $\displaystyle \lim_{x\to c} h(x) = \lim_{x\to c} g(x) = L.$

$$\Rightarrow \lim_{x\to c} f(x) = L$$

Thm 1.9 ① $\displaystyle \lim_{x\to 0} \frac{\sin x}{x} = 1$, ② $\displaystyle \lim_{x\to 0} \frac{1-\cos x}{x} = 0$

pf

①



$$\frac{1}{2}\cdot 1 \cdot \tan\theta \ge \frac{1}{2}\cdot 1^2 \cdot \theta \ge \frac{1}{2}\cdot 1 \cdot \sin\theta$$

Multiply by $\frac{2}{\sin\theta}$

$$\frac{1}{\cos\theta} \ge \frac{\theta}{\sin\theta} \ge 1$$

Taking reciprocals

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1$$

$$\lim_{\theta\to 0} \cos\theta = \cos 0 = 0 = \lim_{x\to 0} 1$$

②

$\cancel{\cos x = \cos\alpha = 2\cos^2\frac{\alpha}{2} - 1} \Leftrightarrow 2\sin^2\frac{x}{2} = 1 - \cos x$

$\cos x = 1 - 2\sin^2\frac{x}{2}$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{2\sin^2\frac{x}{2}}{x} = \lim_{x\to 0} \sin\frac{x}{2} \cdot \frac{\sin\frac{x}{2}}{\frac{x}{2}}$$

$$= \left(\lim_{x\to 0} \sin\frac{x}{2}\right)\cdot\left(\lim_{x\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{2}}\right) = \sin\frac{0}{2}\cdot 1 = 0$$

① $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = \lim\limits_{x \to 0} \dfrac{\sin x}{x} \cdot \dfrac{1}{\cos x} = 1 \cdot 1 = 1$

② $\lim\limits_{x \to 0} \dfrac{\sin 4x}{x} = \lim\limits_{x \to 0} 4 \cdot \dfrac{\sin 4x}{4x} = 4$

1-4

<u>Def</u> f <u>continuous</u> at c if

(i) $f(x)$ defined at $x = c$

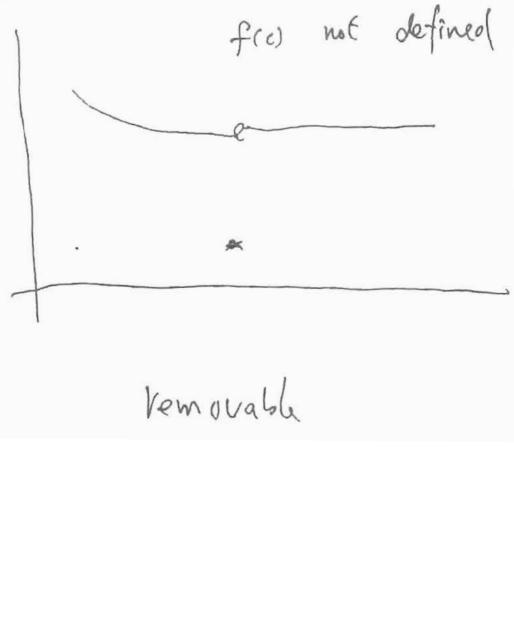
(ii) $\lim\limits_{x \to c} f(x)$ 存在

(iii) $\lim\limits_{x \to c} f(x) = f(c)$

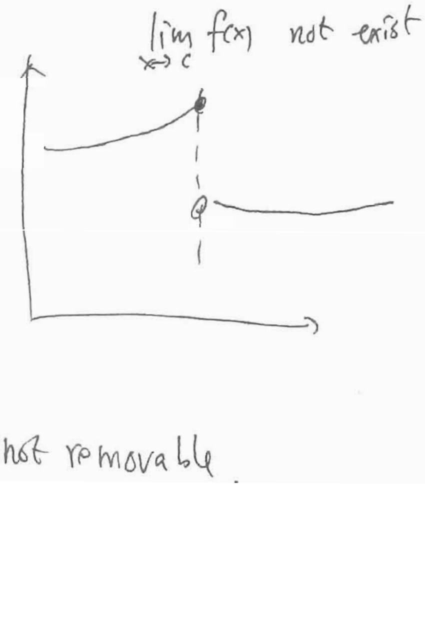
f <u>continuous</u> on (a,b) if $f(x)$ continuous for all $x \in (a,b)$

" $\mathbb{R}$ " " $x \in \mathbb{R}$.

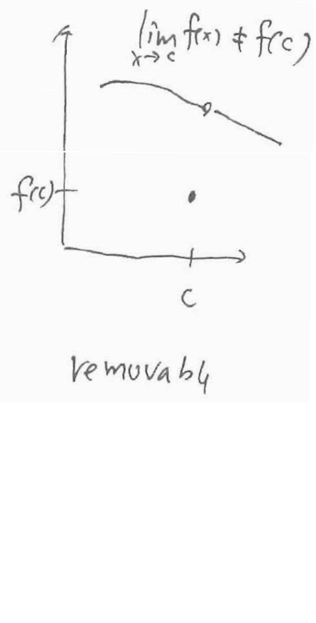
f is called <u>discontinuous</u> if f is not continuous at c.



removable

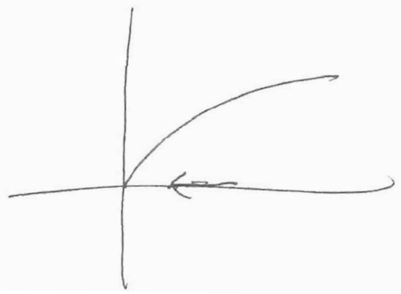


not removable.

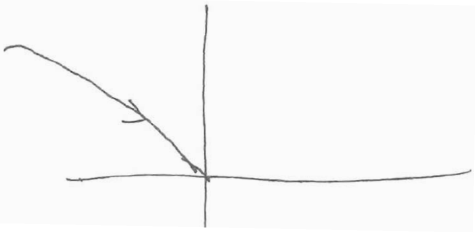


removable

$\lim\limits_{x \to 0^+} \sqrt[2]{x} = 0$ $x \to 0^+ \Leftrightarrow \{ x$



$\lim\limits_{x \to 0^-} \sqrt{-x} = 0$ $x \to 0^- \Leftrightarrow \begin{cases} x < 0 \\ x \to 0 \end{cases}$

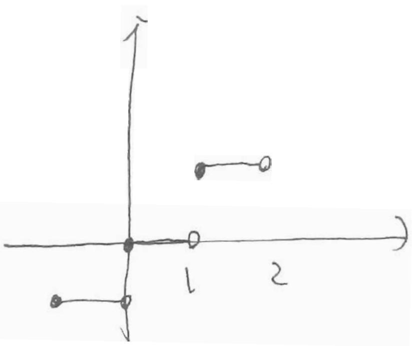


<u>one-side limit</u>

<u>Def</u>: $[\![x]\!]$ = greatest integer less than or equal to x

<u>Ex</u> $[\![25]\!] = 2$

$[\![3]\!] = 3$



$\lim\limits_{x \to 1^+} = 2$

$\lim\limits_{x \to 1^-} = 1$

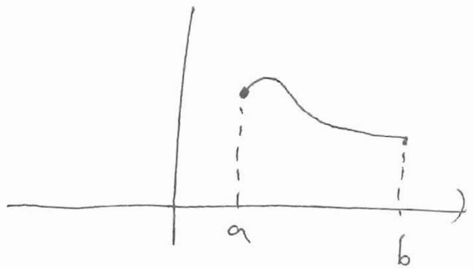
Thm :

$\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x)$

Def : f continuous on $[a, b]$

if ① f continuous on (a, b)

② $\lim_{x \to a^+} f(x) = f(a)$, $\lim_{x \to b^-} f(x) = f(b)$



Ex $f(x) = \sqrt{x}$ continuous on $[0, \infty)$

$g(x) = \sqrt{2-x}$ continuous on $(-\infty, 2]$

Thm 1.11 f, g continuous at $x = c$.

① bf continuous at $x = c$

② $f \pm g$ " "

③ $f \cdot g$ " "

④ $\frac{f}{g}$ if $g(c) \neq 0$.

Thm 1.12 $g(x)$ continuous at c.

$f(x)$ continuous at $f(c)$

$\Rightarrow (f \circ g)(x) = f(g(x))$ continous at $x = c$

∠13

f continuous on $[a,b]$, ~~$f(a) \le f(b)$~~

Suppose that $f(a) \le f(b)$, & $f(a) \le k \le f(b)$

$\Rightarrow$ There exists a $c \in [a,b]$

$f(c) = k$.

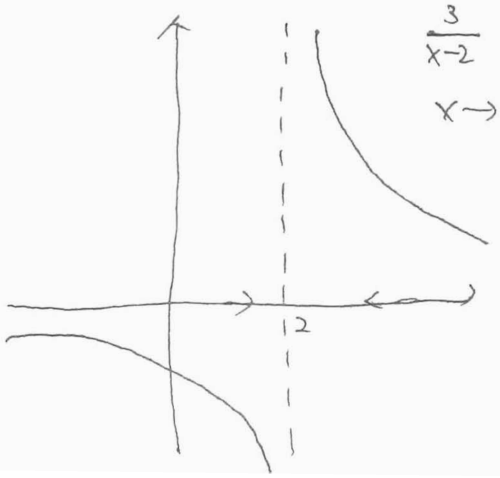
Ex $f(x) = x^3 + 2x - 1$ has a zero on $[0,1]$

pf $f(0) = -1 < 0$

$f(1) = 2 > 0$

$\Rightarrow f(0) \le 0 \le f(1)$

1.5



$\frac{3}{x-2} \to \infty$

$x \to 2^+$

$$\lim_{x \to 2^+} \frac{3}{x-2} = \infty$$

$x \to 2^-$

$\frac{3}{x-2} \to -\infty$

$$\lim_{x \to 2^-} \frac{3}{x-2} = -\infty$$

vertical asymptote

$c - \delta < x < c$

infinite limit from the left

infinite limit from the right

~~ex $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$~~ ex $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$

~~ex $\lim_{x\to}$~~

Thm 1.14 f, g conti on (a, b), $c \in (a, b)$

$f(c) \neq 0$, $g(c) = 0$

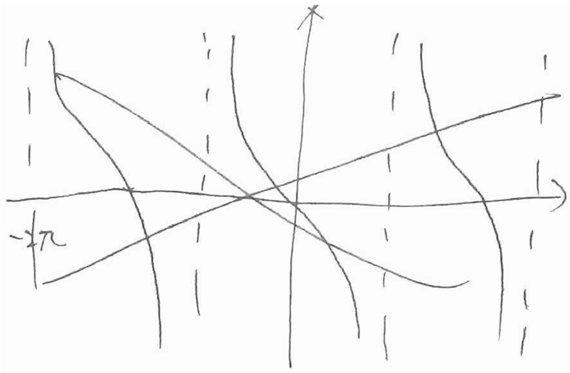
$\Rightarrow$ $x = c$ is a vertical asymptote of $\frac{f(x)}{g(x)}$

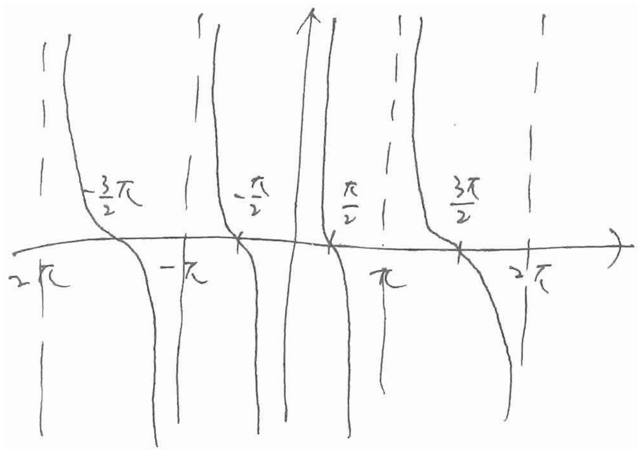
ex 1 $h(x) = \frac{x^2+1}{x^2-1}$

$x = \pm 1$ $\quad x^2 - 1 = 0$ but $x^2+1 \neq 0$

$\Rightarrow$ $x = \pm 1$ vertical asymptotes of $h(x)$

ex 2 $h(x) = \cot x$



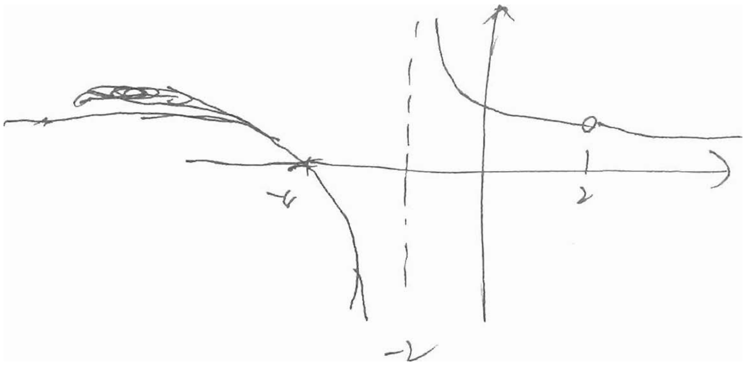


$x = n\pi$ $\quad n \in \mathbb{Z}$ vertical asymptote.

Ex $\quad f(x) = \dfrac{x^2+2x-8}{x^2-4}$

$$x^2-4=0 \Rightarrow x = \pm 2$$

$x \neq 2 \qquad f(x) = \dfrac{(x+4)(x-2)}{(x+2)(x-2)} = \dfrac{x-4}{x+2}$



Ex $\quad \lim_{x\to 0}\left(1+\frac{1}{x^2}\right) = \lim_{x\to 0} 1 + \lim_{x\to 0}\frac{1}{x^2} = \infty$

Ex $\quad \lim_{x\to 1^-} \dfrac{(x^2+1)}{\cot \pi x} = 0$ $\quad$ ~~since $\lim_{x\to 1^-}$~~

since $\lim_{x\to 1^-}(x^2+1) = 2$

$\lim_{x\to 1^-} \cot \pi x = -\infty$

Ex $\quad \lim_{x\to 0^+} 3\cot x = \infty$

since ~~since~~ since $\lim_{x\to 0^+} 3 = 3$, $\lim_{x\to 0^+} \cot x = \infty$