Lecture Notes on Calculus (Class 2012)

Chapter 1

§1.2 Limit of functions:

Intuitively — limit is ~~a prediction~~ the tendency of values of a function $f(x)$ as x approaches some given number c.

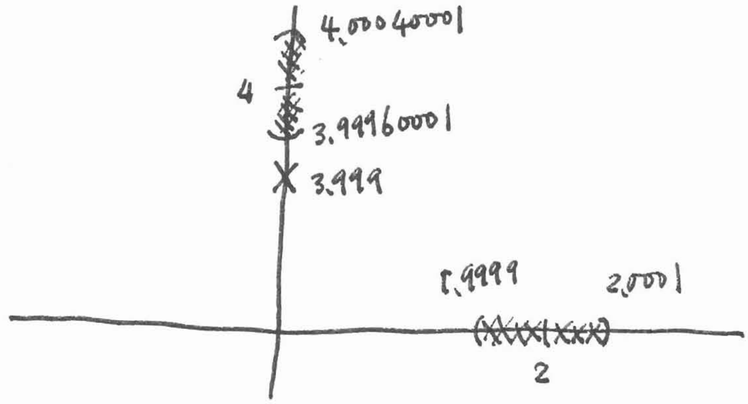
For example: ~~f(x) app~~ The limit of $f(x)=x^2$ as $x \to 2$ is 4 because we observe

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$f(x)$	3.61	3.9601	3.996001	3.99960001		4.00040001	4.004001	4.0401	4.41

The students ~~may~~ are strongly encouraged to compute $f(x)$ when x is much closer to 2

Why 3.999 is not the limit of $f(x)=x^2$ as $x \to 2$?

~~Like~~ From what we have done for any $x \in (1.9999, 2.0001)$
$f(x) \in (3.99960001, 4.00040001)$



Think about it! at what kind of number c we can calculate the limit?

This leads to a rather rigorous def'n of limit.

L is the limit of $f(x)$ as $x \to c$ if
$\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$

Notation: $\lim_{x \to c} f(x) = L$ — L is the limit of f at c or as x approaches

<u>Remark</u>: Even though our example might impress you that ~~lim~~ the limit is a value of f, but finding limit and evaluating value of function are two ~~d~~ absolutely diff. things. For example, $g(x)=\begin{cases} x^2, & x\neq 2 \\ 0, & x=2 \end{cases}$

then $\lim\limits_{x\to 2} g(x) = 4$ still, but $\neq g(2)$

The students should also notice that $\underline{0<|x-c|}$ is required
$\Downarrow$
This is crucial in the sense that $x \neq c$

If $x=c$ is not excluded, then ~~no~~ a lot of functions have no limit at all.

And yes, there are some functions whose limit happens to be the function value ~~of~~, we will come back to this matter later!

<u>Remark</u>: Not every function has limits, for example,

① $f(x)=\begin{cases} 0, & x\in\mathbb{Q} \\ 1, & x\notin\mathbb{Q} \end{cases}$ — Dirichlet's function.
has not limit at any real number c

② $f(x)=\sin\frac{1}{x}$ has no limit ~~at 0~~ as $x\to 0$

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$x\to 0$	$-\frac{2}{11}\pi$	$-\frac{2}{9\pi}$	...
$f(x)$	1	-1	1	-1	1	-1	?	1	-1	...

oscillating

Verify $\lim\limits_{x\to 2} x^2=4$

Let ϵ be any positive number, choose $0<\delta<\min\{1,\frac{\epsilon}{5}\}$

For $x\in(2-\delta,2+\delta)$ but $x\neq 2$, $\left[2-\delta<x<2+\delta \Rightarrow 4-\delta<x+2<4+\delta\right]$ 5

$|x^2-4| = |x+2||x-2| < 5\delta < 5\cdot\frac{\epsilon}{5} = \epsilon.$

§1.3 Properties of Limits:

Thm 1.1: Let b and c be real numbers and let n be a positive integer.

(1) $\lim\limits_{x\to c} b = b$, (2) $\lim\limits_{x\to c} x = c$, (3) $\lim\limits_{x\to c} x^n = c^n$.

Thm 1.2: As assumed in the preceding theorem. Let f and g be functions with the following limits.

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K.$$

(1) $\lim\limits_{x\to c} \left[b f(x)\right] = bL$ (2) $\lim\limits_{x\to c} \left[f(x) \pm g(x)\right] = L \pm K$

(3) $\lim\limits_{x\to c} \left[f(x) g(x)\right] = LK$ (4) $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$

Why needs this?

(5) $\lim\limits_{x\to c} \left(f(x)\right)^n = L^n$.

Thm 1.4 Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for c>0 if n is even

$$\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}.$$

Thm 1.5: If f and g are functions such that $\lim\limits_{x\to c} g(x) = L$ and $\lim\limits_{x\to L} f(x) = f(L)$, then $\lim\limits_{x\to c} f(g(x)) = f(\lim\limits_{x\to c} g(x)) = f(L)$

Thm 1.6: Let [illegible] e one of the six trigonometric functions, and c is be any real number in the domain of f, then

$$\lim_{x\to c} f(x) = f(c).$$

If $\lim\limits_{x\to c} g(x) = K \neq 0$, then $\exists$ a $\delta > 0$ s.t. $g(x) \neq 0$ in $(c-\delta, c+\delta)$, possibly except c.

Thm 1.7. Let c be a real number, and let $f(x)=g(x)$ for all $x \neq c$ in an open interval containing c. If $\lim_{x\to c} g(x)=L$, then the limit of $f(x)$ also exists as $x\to c$

$$\text{and } \lim_{x\to c} f(x)=\lim_{x\to c} g(x)=L.$$

Examples: (1) Find the limit: $\lim_{x\to 1}\frac{x^2+x+2}{x+1}$

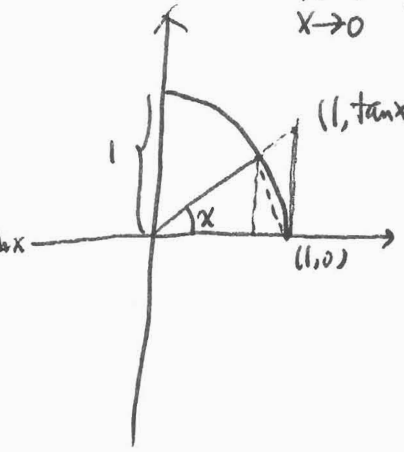
(2) Find the limit: $\lim_{x\to 1}\frac{x^3-1}{x-1}$

(3) Find the limit: $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$.

Thm 1.8: (The Squeeze Thm) If $h(x)\le f(x)\le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if $\lim_{x\to c} h(x)=L=\lim_{x\to c} g(x)$, then $\lim_{x\to c} f(x)$ exists and equals L.

Using Thm 1.8, we can compute

$$\lim_{x\to 0}\frac{\sin x}{x}=1,\ \cancel{\lim_{x\to 0}\frac{1-\cos x}{x}=0}$$



For $x>0$ we have

$$\frac{1}{2}\tan x \ge \frac{1}{2}x \ge \frac{1}{2}\sin x.$$

$$\Rightarrow \frac{1}{\cos x}\ge \frac{x}{\sin x}\ge 1 \Leftarrow$$ also actually valid for $x<0$.

$$\therefore \lim_{x\to 0}\frac{x}{\sin x}=1. \Rightarrow \lim_{x\to 0}\frac{\sin x}{x}=1.$$

~~In the~~ ~~From the preceding proof we witness~~ a

§1.4 Def'n: A function f is continuous at c if

(1) f is defined at c

(2) $\lim_{x \to c} f(x)$ exists

(3) $\lim_{x \to c} f(x) = f(c)$.

Def'n: A function is continuous on an open interval (a, b) if it is continuous at each point in the interval.

Def'n: Let I be an open interval containing c. Let f be a function defined on I, except possibly at c. If f is not continuous at c, then it is called that f has a discontinuity at c.

According to the def'n, f having a discontinuity at c suggests ~~indicate~~

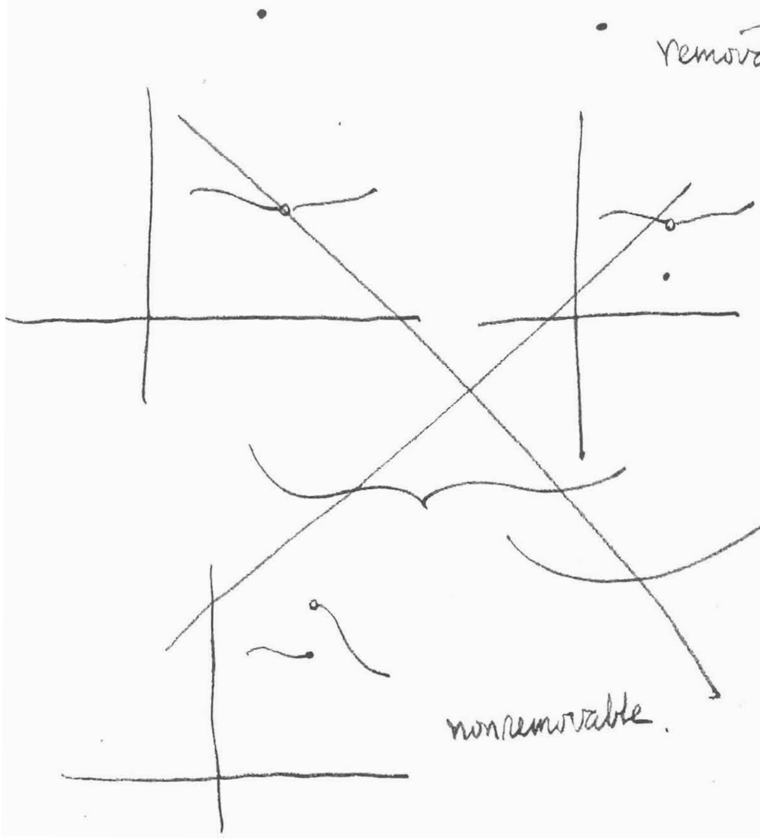
(1) f is not defined at c, (2) f has no limit at c

(3) $f(c) \neq \lim_{x \to c} f(x)$

nonremovable

vs.

removable — f(c) can be appropriately re-defined so that $f(c) = \lim_{x \to c} f(x)$ (provided $\lim_{x \to c} f(x)$ exists in (1) & (3))



nonremovable.

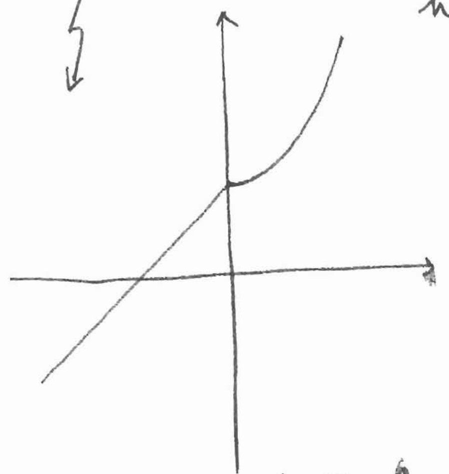
explain examples on the next page!

Examples: (1) $f(x)=\frac{1}{x}$, 0 is the only discontinuity (non removable)

(2) $g(x)=\frac{x^2-1}{x-1}$ $(=x+1)$: 1 is the only discontinuity (removable)

(3) $h(x)=\begin{cases} x+1, & x\le 0 \\ x^2+1, & x>0 \end{cases}$ and (4) $y=\sin x$

have no discontinuities.



Induces one-sided limit;

the limit from the right, $\lim_{x\to c+} f(x)=L$ means

$\forall \epsilon>0, \exists \delta>0$ s.t. $|f(x)-L|<\epsilon$ whenever $0<x-c<\delta$.

the limit from the left, $\lim_{x\to c-} f(x)=L$ means

$\forall \epsilon>0\ \exists \delta>0$ s.t. $|f(x)-L|<\epsilon$ whenever $-\delta<x-c<0$

So, it is easy to see

Thm 1.10: Let f be a function and let c and L be real numbers. The limit of $f(x)$ as $x\to c$ is L if and only if

$$\lim_{x\to c+} f(x)=L \text{ and } \lim_{x\to c-} f(x)=L.$$

Defn: A function f is continuous on the closed interval $[a,b]$ if it is continuous on (a,b) and $\underbrace{\lim_{x\to a+} f(x)=f(a)}_{\text{also called } f \text{ is continuous from the right at } a}$ and $\underbrace{\lim_{x\to b^-} f(x)=f(b)}_{\dots}$

Example: $f(x)=\sqrt{1-x^2}$ is continuous on its domain!
(DO NOT NEED TO DISCUSS IT IN CLASS)

Theorem 1.11: If b is a real number and f and g are continuous at $x=c$, then the following functions are also continuous at c.

(1) Scalar multiple: bf
(2) Sum and difference: $f \pm g$
(3) Product: fg
(4) Quotient: f/g, if $g(c) \neq 0$

Obviously, these are immediate deductions of Thm 1.2.

Theorem 1.12: If g is continuous at c and f is continuous at $g(c)$, then the composition function $f \circ g$, defined by $(f \circ g)(x) = f(g(x))$, is continuous at c.

Thm 1.13: (Intermediate Value Theorem)
If f is continuous on the closed interval $[a,b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ s.t. $f(c)=k$.
(Probably, the most familiar (to the students) of the applications of this theorem is to guarantee the existence of zeros)
e.g. x^3+2x-1 has a zero between 0 and 1.

Working Example:

Describe the interval(s) on which each function is continuous

(a) $f(x)=\begin{cases} \sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$ (b) $g(x)=\begin{cases} x\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0. \end{cases}$

(This may be skipped)

§1.5 As we look at the graph of $f(x) = \frac{1}{x}$, it seems possible to describe the tendency of the values where x is ~~very large~~ approaches zero. This leads to

Def'n: Let f be a function defined at every real number in some open interval containing c (except possibly at c itself).

$\lim_{x \to c} f(x) = \infty$ means $\forall M > 0, \exists \delta > 0$ s.t. $f(x) > M$ whenever $0 < |x-c| < \delta$.

$\lim_{x \to c} f(x) = -\infty$ means $\forall N < 0\ \exists \delta > 0$ s.t. $f(x) < N$ whenever $0 < |x-c| < \delta$.

Either one is called that f has an infinite limit as $x \to c$.

The students are strongly encouraged to write down the meanings of $\lim_{x \to c+} f(x) = \infty$, $\lim_{x \to c+} f(x) = -\infty$, $\lim_{x \to c-} f(x) = \infty$ and $\lim_{x \to c-} f(x) = -\infty$

Remark: $\infty, -\infty$ are NOT numbers $\lim_{x \to c} f(x) = \infty$ doesn't mean that the values of f ~~cluster~~ be very close to some number called ∞ but be larger and larger. So restrictly speaking, $\lim_{x \to c} f(x) = \infty$ indicates that f has no limit at c.

Def'n: If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a vertical asymptote of the graph of f.

Examples: $\frac{1}{x-1}$, $\frac{1}{(x-1)^2}$, $\tan x$, $\sec x$, $\cdots$ (Don't need to work)

Thm 1.14: Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function given by

$$h(x) = \frac{f(x)}{g(x)}$$

has a vertical asymptote at $x = c$.

Working Examples:

Determine all vertical asymptotes of the graph of $f(x) = \frac{x^2+2x-8}{x^2-4}$ $\quad (x = -2)$

$$\left(\lim_{x \to -2^+} f(x) = \infty, \ \lim_{x \to -2^-} f(x) = -\infty \right)$$

Thm 1.15 Let c and L be real numbers and let f and g be functions s.t. $\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} g(x) = L$

(1) $\lim_{x \to c} [f(x) \pm g(x)] = \infty$

(2) $\lim_{x \to c} [f(x)g(x)] = \infty$, if $L > 0$
$-\infty$, if $L < 0$
?, if $L = 0$

(3) $\lim_{x \to c} \frac{g(x)}{f(x)} = 0$.

[Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as $x \to c$ is $-\infty$]

Think about it! Look at the graph of $f(x) = \frac{1}{x}$ again.

Think: How do you describe ~~the~~ horizontal asymptotes?

You may also look at the graph of $x^2 - y^2 = 1$.

Think: How do you describe the slant asymptotes?