§15.4 Green's Thm.

A plane region R is <u>simply connected</u> if its bdy consists of a simple closed curve.

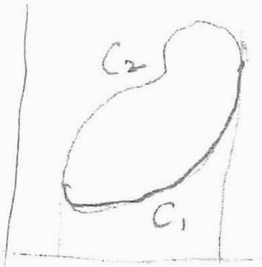
<u>Thm 15.8</u> (Green's Thm)

R: simply connected region with piecewise smooth bdy C, oriented counterclockwise (R lies to the left of C).

If M and N have cts partials in an open region containing R, then

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\overset{dA}{\cancel{dx\,dy}} = \int_C M\,dx + N\,dy$$

<u>Pf.</u> We give a proof when the region R is both vertical and horizontal simple as shown in the figure.



$$\int_C M\,dx = \int_{C_1} M\,dx + \int_{C_2} M\,dx$$

$$= \int_a^b M(x, f_1(x))\,dx + \int_b^a M(x, f_2(x))\,dx$$

$$= \int_a^b M(x, f_1(x)) - M(x, f_2(x))\,dx$$

$$\iint_R \frac{\partial M}{\partial y}\,dA = \int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\,dy\,dx = \int_a^b M(x,y)\Big|_{f_1(x)}^{f_2(x)}\,dx$$

$$= \int_a^b \big(M(x, f_2(x)) - M(x, f_1(x))\big)\,dx$$

$$\Rightarrow \int_C M\,dx = -\iint_R \frac{\partial M}{\partial y}\,dA \qquad *$$

<u>Examples</u> 1. 3. 4.

<u>Thm 15.9</u> In the special case we have

$$A^{(R)} = \frac{1}{2}\int_C x\,dy - y\,dx$$ for a simple closed region R with bdy C.

<u>Examples</u> 5. 6

Similarly, $\overrightarrow{MQ} \approx \frac{\partial x}{\partial v}\Delta v\,\vec{i} + \frac{\partial y}{\partial v}\Delta v\,\vec{j}$. So

$$\|\overrightarrow{MN} \times \overrightarrow{MQ}\| = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u & 0 \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v & 0 \end{matrix}\right| = \left|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{matrix}\right| \Delta u \Delta v$$

So $dA = dx\,dy = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$. ※.

Example 3, 4.

Ch15 Vector Analysis.

§15.1 Vector Fields.

Df R be a region in $\mathbb{R}^2$ or Q a region in $\mathbb{R}^3$.

A vector field over R or Q is a function

$F(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ over R

or $F(x,y,z) = M(x,y,z)\vec{i} + N(x,y,z)\vec{j} + P(x,y,z)\vec{k}$ over Q.

A typical example of a v.f. is the gradient of a function, namely for $f(x,y)$ or $f(x,y,z)$, let

$F(x,y) = \nabla f = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$,

or $= f_x(x,y,z)\vec{i} + f_y(x,y,z)\vec{j} + f_z(x,y,z)\vec{k}$

In this case, we say the v.f. F is <u>conservative</u> and f is called the <u>potential function</u> for F.

<u>Thm 15.1</u> A v.f. $F(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ with its first partials defined on a plane region R is conservative iff

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Pf "⇒" $M = f_x(x,y)$, $N = f_y(x,y)$ for some f s.t. $F = \nabla f$

then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Examples 5, 6.

Curl of a v.f. in space

Def. The curl of a vector field $F(x,y,z) = M\vec{i} + N\vec{j} + P\vec{k}$ is

$$\text{curl } F(x,y,z) = \nabla \times F(x,y,z)$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\vec{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\vec{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Thm 15.2 Suppose that the v.f. $F(x,y,z) = M\vec{i} + N\vec{j} + P\vec{k}$ has cts first partials for M, N and P, then F is conservative iff $\text{curl } F(x,y,z) = 0$.

Example 8

Divergence of a v.f.

The divergence of a vector field $F(x,y) = M\vec{i} + N\vec{j}$

or $F(x,y,z) = M\vec{i} + N\vec{j} + P\vec{k}$

is defined to be $\text{div } F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$

or $= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$.

Thm 15.3 If $F(x,y,z) = M\vec{i} + N\vec{j} + P\vec{k}$ and M, N and P have cts first partials, then

$$\text{div}(\text{curl } F) = 0$$

§15.2 Line integrals.

A curve $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$ on the interval $[a, b]$

or $= x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$

is piecewise smooth if $[a, b]$ can be partitioned into a finite number of subintervals, on each of which $\vec{r}(t)$ is smooth.

Def (line integral)
If f is defined on a region containing a smooth curve C of finite length then the line integral of f along C is given by

$$\int_C f(x, y)\, ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\, ds$$

or $$\int_C f(x, y, z)\, ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\, ds$$

where the partition is taken on C.

Thm 15.4 In the plane region case, if C is given by
$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$, $a \le t \le b$, then

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

In the space case, if C is given by
$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$, $a \le t \le b$, then

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

Examples 3. 4.

Def $\vec{F}$ cts v.f. defined on a smooth curve C given by $\vec{r}(t)$
$a \le t \le b$. The line integral of $\vec{F}$ on C is given by

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T}\, ds = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t)\, dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b (M\vec{i} + N\vec{j}) \cdot (x'(t)\,\vec{i} + y'(t)\,\vec{j})\, dt = \int_a^b \left(M \frac{dx}{dt} + N \frac{dy}{dt}\right) dt = \int_C M\,dx + N\,dy.$$

differential form
representation of line integral

Examples 8 9

§15.3 Conservative v.f. and independence of path.

<u>Thm 15.5</u> C: piecewise smooth curve in an open region R given by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ $a \le t \le b$.

If $F(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ is conservative in R and M and N are cts in R, then

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(x(b), y(b)) - f(x(a), y(a))$$

where f is the potential of F i.e. $F = \nabla f$.

Pf. Suppose that C is smooth.

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}\cdot \frac{d\vec{r}}{dt}\, dt = \int_a^b \left(f_x(x,y)\frac{dx}{dt} + f_y(x,y)\frac{dy}{dt}\right) dt$$

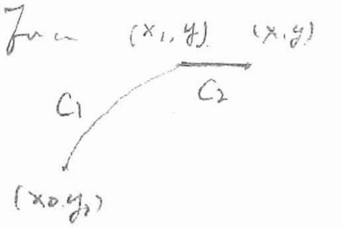
$$= \int_a^b \frac{d}{dt} f(x(t), y(t))\, dt = f(x(b), y(b)) - f(x(a), y(a)).$$

<u>Thm 15.6</u> If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path, then F is conservative.

Pf. For any $(x,y) \in R$, take a curve C connectin (x_0,y_0) and (x,y). Take $(x_0,y_0) \in R$

Define the function $f(x,y) = \int_C \vec{F}\cdot dr = \int_C M\,dx + N\,dy$.

Since $f(x,y)$ is independent of C, we may take C to be of the form



$$f(x,y) = \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy$$

The first integral does NOT dependent on x.
$dy = 0$ in the second integral

$$f(x,y) = g(y) + \int M\,dx \rightarrow \frac{\partial f}{\partial x} = M$$

Same reason $\frac{\partial f}{\partial y} = N$. Hence f is a potential of F.

#

<u>Example 2.3</u>