Ch 12 Vector-valued functions

§12.1 Vector-valued functions

Def A func. of the form

$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ plane curve

or $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ space curve

is called a vector-valued function.

$x(t), y(t), z(t)$ are real-valued functions.

We may also write

$\vec{r}(t) = (x(t), y(t))$ or $\vec{r}(t) = (x(t), y(t), z(t))$

In general, two different curves may have the same graph.

For example $\vec{r}(t) = (\cos t, \sin t)$ $\vec{r}(t) = (\sin t, \cos t)$.

Def (Limit of a vector-valued function)

1. If $\vec{r}(t) = (f(t), g(t))$, then

$$\lim_{t \to t_0} \vec{r}(t) = \left(\lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t) \right)$$

2. If $\vec{r}(t) = (f(t), g(t), h(t))$, then

$$\lim_{t \to t_0} \vec{r}(t) = \left(\lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t) \right)$$

We can check that if $\vec{r}(t) \to \vec{L}$ as $t \to t_0$ then

$\|\vec{r}(t) - \vec{L}\| \to 0$ as $t \to t_0$

Def A curve (plane or space) $\vec{r}(t)$ is cts at $t = t_0$ if

$$\lim_{t \to t_0} \vec{r}(t) = \vec{r}(t_0).$$

$\vec{r}(t)$ is cts on $t \in I$ if it is cts at each point of I.

— Note that a vector-value function $\vec{r}(t)$ is cts iff each component is cts.

§12.2 Differentiation and integration

<u>Def</u> The derivative of a vector-valued function $\vec{r}(t)$ is defined by

$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

If $\vec{r}\,'(c)$ exists, $\vec{r}(t)$ is differentiable at c.

<u>Thm 12.1</u> 1. If $\vec{r}(t) = (f(t), g(t))$, then $\vec{r}\,'(t) = (f'(t), g'(t))$

2. If $\vec{r}(t) = (f(t), g(t), h(t))$, then $\vec{r}\,'(t) = (f'(t), g'(t), h'(t))$

- $\vec{r}\,'(t)$ exists iff it components are differentiable
- $\vec{r}(t) = (x(t), y(t), z(t))$ is <u>smooth</u> on an open interval I if $x'(t)$, $y'(t)$, $z'(t)$ are cts on I and $\vec{r}\,'(t) \neq \vec{0}$ on I.

<u>Def</u> 1. If $\vec{r}(t) = (f(t), g(t))$, where f and g are cts on $[a,b]$, then the integration of $\vec{r}(t)$ is

$$\int \vec{r}(t)\,dt = \left(\int f(t)\,dt,\ \int g(t)\,dt\right)$$

2. If $\vec{r}(t) = (f(t), g(t), h(t))$, where f, g, h are cts on $[a,b]$, then

$$\int \vec{r}(t)\,dt = \left(\int f(t)\,dt,\ \int g(t)\,dt,\ \int h(t)\,dt\right)$$

§12.4 Tangent and normal vectors

~~Def~~ C: smooth curve represented by $\vec{r}(t)$ on the interval I. The <u>unit tangent vector</u> $\vec{T}(t)$ at t is defined by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \quad \vec{r}'(t) \neq 0.$$

$\vec{T}(t)$ like $\vec{r}(t)$ depends on the orientation

<u>Example 2</u>

Since $T(t)$ is a unit vector we have,

$$\vec{T}(t) \cdot \vec{T}(t) = \|\vec{T}(t)\|^2 = 1.$$

Differentiating gives $\vec{T}(t) \cdot \vec{T}'(t) = 0$. i.e. $\vec{T}'(t) \perp \vec{T}(t)$

~~Def~~ For a smooth curve C defined by $\vec{r}(t)$, ~~define the~~ if $\vec{T}'(t) \neq 0$, define the <u>principle unit normal</u> vector at t to be

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|}$$

<u>Example 4</u>

§12.5 Arc length and curvature.

Arc length.

C: smooth curve given by $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ on an interval $I = [a, b]$, then the arc length of C is

$$s = \int_a^b \|\vec{r}'(t)\|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

<u>Example 2</u>

~~Def~~ For $t \in [a, b)$, we can define the arc length func.

$$s(t) \doteq \int_a^t \|\vec{r}'(u)\| du = \int_a^t \sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2}\, du$$

Note that we have $\frac{ds}{dt} = \|\vec{r}'(t)\|$ or $ds = \|\vec{r}'(t)\| dt$

Call s the arc length parameter

Note that in the arc length parameter, $\|\vec{r}'(s)\| = 1$.

Curvature

Def. C: smooth curve parametrized by arc length parameter s. Then the curvature κ at s is given by

$$\kappa = \left\|\frac{d\vec{T}}{ds}\right\| = \|\vec{T}'(s)\|.$$

Example 4 Example 5

Thm 12.8 Formula for curvature

C: smooth curve given by $\vec{r}(t)$. Then the curvature κ of C at t is given by

$$\kappa = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

If the curve is given by $(x, f(x)) = \vec{r}(x)$, $f(x) = y$, then

$$\kappa = \frac{y''}{(1+(y')^2)^{3/2}}$$

pf. $\vec{T} = \frac{\vec{r}'}{\|\vec{r}'\|}$ $\|\vec{r}'\| = \frac{ds}{dt}$ $\vec{r}' = \|\vec{r}'\|\vec{T} = \frac{ds}{dt}\vec{T}$

$$\vec{r}'' = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T}'$$

$$\vec{r}' \times \vec{r}'' = \left(\frac{ds}{dt}\right)^2 (\vec{T} \times \vec{T}')$$

$$\|\vec{r}' \times \vec{r}''\| = \left(\frac{ds}{dt}\right)^2 \|\vec{T}\|\|\vec{T}'\| = \left(\frac{ds}{dt}\right)^2 \|\vec{T}'\|$$

So $$\kappa = \frac{\|\vec{T}'\|}{\|\vec{r}'\|} = \frac{\|\vec{r}' \times \vec{r}''\|}{\|\vec{r}'\|^3}$$