Ch 11 Vectors and the geometry of space.

§§ 11.1, 11.2 Vectors.

A vector $\vec{v} = \langle v_1, v_2 \rangle$ if it is on the plane $\mathbb{R}^2$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$ - - in the space $\mathbb{R}^3$.

~~length of~~ Two vectors $\underset{(u_1, u_2)}{\vec{u}}$, $\underset{\langle v_1, v_2 \rangle}{\vec{v}}$ and $\alpha \in \mathbb{R}$, we set

$$\vec{u} + \alpha\vec{v} = (u_1 + \alpha v_1, u_2 + \alpha v_2)$$

The length of a vector $\vec{u} = (u_1, u_2)$ is

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2}.$$

Thm vector operations

If $\vec{u}, \vec{v}, \vec{w}$ are vectors, $c, d \in \mathbb{R}$,

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ $\qquad \|c\vec{v}\| = |c|\,\|\vec{v}\|$.
3. $\vec{u} + \vec{0} = \vec{u}$
4. $\vec{u} + (-\vec{u}) = \vec{0}$
5. $c(d\vec{u}) = (cd)\vec{u}$
6. $(c+d)\vec{u} = c\vec{u} + d\vec{u}$
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
8. $1\vec{u} = \vec{u}$, $0\vec{u} = \vec{0}$.

§ 11.3. Dot product

Def $\vec{u}, \vec{v}$ ~~are~~ vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, then we define the dot ~~of~~ product of $\vec{u}$ and $\vec{v}$ to be

(i) If $\vec{u} = (u_1, u_2)$ $\vec{v} = \langle v_1, v_2 \rangle$, then

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2$$

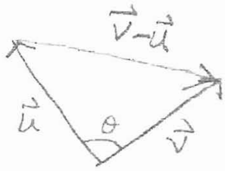
(ii) If $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = \langle v_1, v_2, v_3 \rangle$, Then

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Properties $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, $c, d \in \mathbb{R}$, then

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $c(\vec{u} \cdot \vec{v}) = c\vec{u} \cdot \vec{v} = \vec{u} \cdot c\vec{v}$.
4. $\vec{0} \cdot \vec{v} = 0$
5. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

The angle between two vectors $\vec{u}, \vec{v}$ is the angle $0 \le \theta \le \pi$ between them



Thm 11.5 If $\vec{u} \neq 0$, $\vec{v} \neq 0$, then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$$

Pf $\|\vec{v}-\vec{u}\|^2 = \|u\|^2 + \|v\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta$

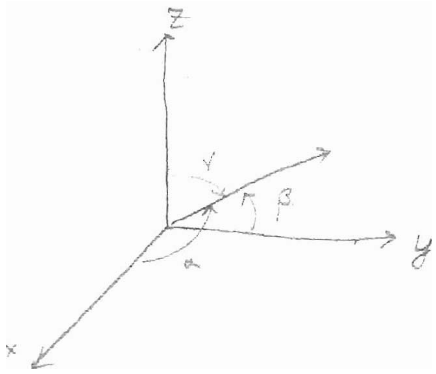
$$\| \qquad\qquad\qquad\qquad \vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

$\|\vec{v}\|^2 - 2\vec{v}\cdot\vec{u} + \|\vec{u}\|^2$.

Def $\vec{u}, \vec{v}$ are said to be orthogonal if $\vec{u}\cdot\vec{v} = 0$.

Direction cosines

$$\vec{v} = (v_1, v_2, v_3)$$



$$\cos\alpha = \frac{v_1}{\|\vec{v}\|}$$

Projection The projection of $\vec{u}$ on $\vec{v}$ is given by

$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{\langle \vec{u}, \vec{v}\rangle}{\|\vec{v}\|^2}\,\vec{v}.$$

§11.4 Cross product of two vectors in space.

<u>Def.</u> $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$, $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ vectors in space

The <u>cross product</u> of $\vec{u}$ and $\vec{v}$ is the vector.

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}.$$

<u>Thm 11.9</u> Alg. properties of the cross product

$\vec{u}, \vec{v}, \vec{w}$ be vectors in space, c a scalar

1. $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$
2. $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$.
3. $c(\vec{u}\times\vec{v}) = (c\vec{u})\times\vec{v} = \vec{u}\times c\vec{v}$
4. $\vec{u}\times\vec{0} = \vec{0}\times\vec{u} = \vec{0}$
5. $\vec{u}\times\vec{u} = \vec{0}$
6. $\vec{u}\cdot(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\cdot\vec{w}$

<u>Thm 11.8</u> $\vec{u}, \vec{v}$ non-zero vectors in space, θ angle between $\vec{u}$ and $\vec{v}$

1. $\vec{u}\times\vec{v} \perp \vec{u}$ and $\vec{v}$
2. $\|\vec{u}\times\vec{v}\| = \|u\|\,\|v\|\sin\theta$.
3. $\vec{u}\times\vec{v} = 0$ iff $\vec{u} = \lambda\vec{v}$ for some scalar
4. $\|\vec{u}\times\vec{v}\|$ = area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.

Pf. $\|\vec{u}\|\|\vec{v}\|\sin\theta = \|\vec{u}\|\|\vec{v}\|\sqrt{1-\cos^2\theta} = \|\vec{u}\|\|\vec{v}\|\sqrt{1-\dfrac{(\vec{u}\cdot\vec{v})^2}{\|u\|^2\|v\|^2}}$

$$= \sqrt{\|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}\cdot\vec{v})^2} = \sqrt{(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)}$$

$$= \sqrt{(u_2v_3-u_3v_2)^2 + (u_1v_3-u_3v_1)^2 + (u_1v_2-u_2v_1)^2}$$

$$= \|\vec{u}\times\vec{v}\|$$

<u>Thm 11.9</u> (Triple scalar product)

For $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$, $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$

the triple scalar product is given by

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

It has the meaning that $|\vec{u}\cdot(\vec{v}\times\vec{w})| = V$ is the volume of the parallelepiped spanned by $\vec{u}, \vec{v}$ and $\vec{w}$.

Pf. $\|\vec{v}\times\vec{w}\|$ = area of the base

$\|\mathrm{proj}_{\vec{v}\times\vec{w}}\vec{u}\|$ = height of the parallelepiped

$$V = \|\mathrm{proj}_{\vec{v}\times\vec{w}}\vec{u}\|\,\|\vec{v}\times\vec{w}\| = \frac{|\vec{u}\cdot(\vec{v}\times\vec{w})|}{\|\vec{v}\times\vec{w}\|}\|\vec{v}\times\vec{w}\| = |\vec{u}\cdot(\vec{v}\times\vec{w})|.$$

§11.5 Lines and planes in space.

Thm 11.1 (parametric eq. of a line in space)
A line L parallel to the vector $\vec{v} = (a, b, c)$ passing through the pt $P(x_1, y_1, z_1)$ is represented by the parametric equation:

$$x = x_1 + at, \quad y = y_1 + bt, \quad z = z_1 + ct.$$

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

Thm 11.2 (standard eq of a plane in space) with normal vector $n = (a,$
The plane containing the pt (x_1, y_1, z_1)
is represented by the equation

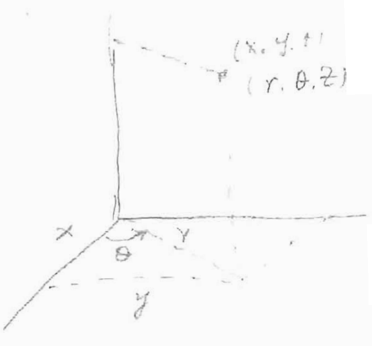
$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

~~The projection of $\vec{u}$ on $\vec{v}$ is given by~~

$$\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$$

§11.7 Cylindrical and spherical coordinates

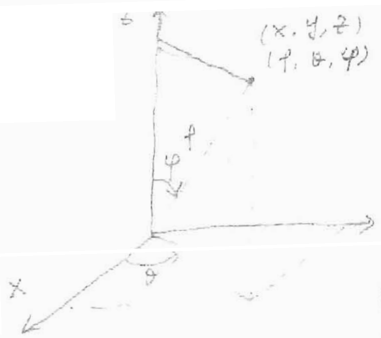
Cylindrical



$x = r\cos\theta \quad y = r\sin\theta$

$r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x}$

Spherical



$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi. \end{cases}$$

$$\begin{cases} \rho^2 = x^2 + y^2 + z^2 \\ \tan\theta = \frac{y}{x} \\ \varphi = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} \end{cases}$$

spherical $\Rightarrow$ cylindrical

$r^2 = \rho^2 \sin^2\varphi \qquad \theta = \theta. \qquad z = \rho\cos\varphi.$

cylindrical $\Rightarrow$ spherical

$\rho = \sqrt{r^2 + z^2} \qquad \theta = \theta \qquad \varphi = \arccos \frac{z}{\sqrt{r^2 + z^2}}$