Ch 10 Conics

§10.1 Conics

Thm 10.1 (parabola)

The standard form of the eq. of a parabola with vertex (h,k) and directrix $y = k-p$ is

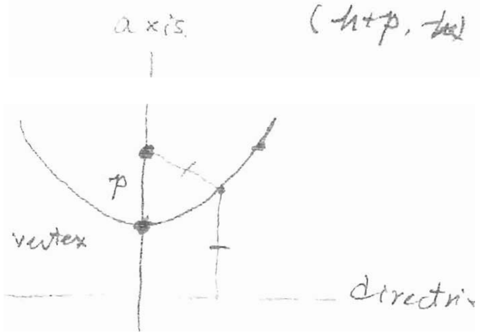
$$(x-h)^2 = 4p(y-k) \qquad \text{vertical axis}$$

For directrix $x = h-p$, the eq is

$$(y-k)^2 = 4p(x-h) \qquad \text{horizontal axis}$$

Focus $(h, k+p)$ vertical

$(h+p, k)$ horizontal



Thm 10.3 (ellipse)

The standard form for an ellipse with center (h,k) and major and minor axes of length $2a$ and $2b$, $a>b$ is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad \text{major is horizontal}$$

or

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \qquad \text{major is vertical}$$

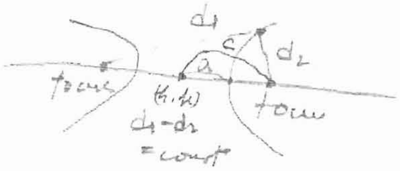
The foci lie on the major axis, c units from the center with $c^2 = a^2 - b^2$.

Set $e = \frac{c}{a}$ eccentricity.

Thm 10.4 (hyperbola) center (h,k)

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1.$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$



Foci satisfying

$c^2 = a^2 + b^2$

asymptotes of a hyperbola

$y = k + \frac{b}{a}(x-h)$ $\qquad$ $y = k - \frac{b}{a}(x-h)$

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§10.2 plane curves and parametric equations

Def f, g cts functions in t on an interval I, then the eqs

$$x = f(t), \quad y = g(t)$$

are parametric eqs. t: parameter.

The graph of the parametric eqs is called a plane curve

Ex 2.3 p 711

§10.3 parametric eq and calculus

Thm 10.7 If a smooth curve is given by the eqs $x = f(t)$, $y = g(t)$, then the solpe at (x, y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \text{if } \frac{dx}{dt} \neq 0.$$

examples 1, 2

Arc length $\quad x = f(t), \; y = g(t) \quad a \le t \le b, \quad \frac{dx}{dt} = f'(t) > 0.$

$$S = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\,\cancel{dt} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt$$

Thm 10.9 (Area of a surface of revolution)

C smooth curve given by $x = f(t)$, $y = g(t)$ which does NOT cross itself on $a \le t \le b$.

Then the surface of revolution formed by revolving C about the coordinate axes is

$$S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad x\text{-axis and if } g \ge 0.$$

$$S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad y\text{-axis and } f \ge 0.$$

§10.4 Polar coordinates and polar graphs

(r,θ) polar coordinate.

Not unique $(r,\theta) = (r,\ \theta+2n\pi) = (-r,\ \theta+(2n+1)\pi)$.

The pole is represented by $(0,\theta)$ for any θ.

<u>Thm 10.10</u>

$$x = r\cos\theta \qquad \tan\theta = \frac{y}{x}$$

$$y = r\sin\theta \qquad r^2 = x^2+y^2$$

<u>Thm 10.11</u> If $f(\theta)$ is differentiable, then the slope to the graph $r=f(\theta)$ at the point (r,θ) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}. \qquad \text{If } \frac{dx}{d\theta} \neq 0. \text{ at } (r,\theta).$$

<u>Example 5</u> p 733
6.

Suppose the graph of $r=f(\theta)$ passing through the pole at $\theta=\alpha$ and $f'(\alpha)\neq 0$, then

$$\frac{dy}{dx} = \frac{f'(\alpha)\sin\alpha + f(\alpha)\cos\alpha}{f'(\alpha)\cos\alpha - f(\alpha)\sin\alpha} = \frac{f'(\alpha)\sin\alpha + 0}{f'(\alpha)\cos\alpha + 0} = \tan\alpha.$$

This implies $\theta=\alpha$ is tangent to the graph at the pole, $(0,\alpha)$.

§10.5 Area and arc length in polar coordinates.

<u>Thm 10.13</u> If f is cts and nonnegative on $[\alpha,\beta]$, $0<\beta-\alpha\le 2\pi$ then the area of the region bdd by $r=f(\theta)$ between $\theta=\alpha$, $\theta=\beta$ is

$$A = \frac{1}{2}\int_\alpha^\beta (f(\theta))^2\,d\theta = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta.$$

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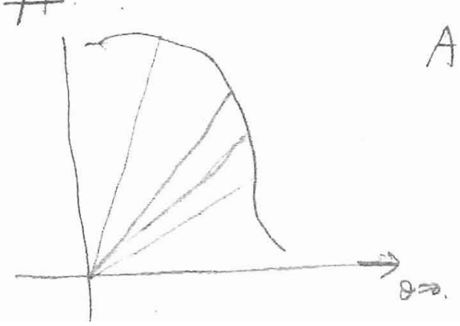

$$A = \lim_{n\to\infty} \frac{1}{2}\sum_{i=1}^{n} f(\theta_i)^2\,\Delta\theta$$

<u>Example 1</u>

Arc length in polar form

Thm 10.14 f ~~cts~~ with cts derivative on $\alpha \le \theta \le \beta$, then the length of the graph $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Example 4

Thm 10.15 Surface area

$r = f(\theta)$, $\alpha \le \theta \le \beta$.

$$S = 2\pi \int_\alpha^\beta f(\theta) \sin\theta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$ rotate about the polar axis

$$S = 2\pi \int_\alpha^\beta f(\theta) \cos\theta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$$ about the axis $\theta = \pi/2$

Example 5