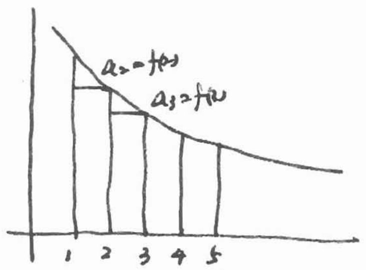
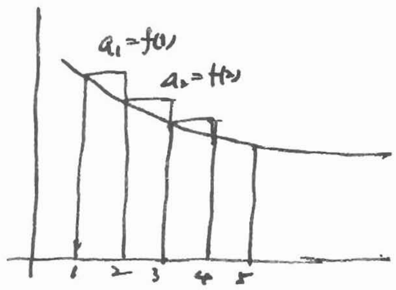
§9.3 The integral test and p-series.

<u>Integral test</u> : convergence test for series with positive terms.

<u>Thm 9.10</u>
If f is positive, cts and decreasing for $x \geq 1$, and $a_n = f(n)$, then
$\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)dx$ either both converge or both diverge.

pf. Consider the two diagrams





$$\sum_{i=2}^{n} f(i) \leq \int_1^n f(x)\,dx \leq \sum_{i=1}^{n-1} f(i)$$

$$S_n - f(1) \leq \int_1^n f(x)\,dx \leq S_{n-1}$$

If $\int_1^{\infty} f(x)dx$ converges, then S_n converges.

S_{n-1} converges implies $\int_1^n f(x)dx$ converges.

<u>Examples</u> 1. $\sum \frac{n}{n^2+1}$. $f(x) = \frac{x}{x^2+1}$ > 0, positive. $f'(x) = \frac{-x^2+1}{(x^2+1)^2} < 0$.

2 $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $\int_1^{\infty} \frac{1}{x^2+1}dx = \lim_{b\to\infty} \left(\arctan x \Big|_1^b\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$.

<u>p-series</u> The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a p-series.
$p=1$. Harmonic series.

<u>Thm 9.11</u> The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

(i) converges if $p > 1$

(ii) diverge if $0 < p \leq 1$.

<u>Examples</u> 3 $\sum \frac{1}{n}$, 4. $\sum \frac{1}{n^2}$.

5. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ integral test.

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ ...

§9.4 Comparison of series.

Thm 9.12 (direct comparison test)

Let $0 < a_n \le b_n$ for all n:

1 If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2 If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Pf. 1. Let $\sum_{n=1}^{\infty} b_n = L$, and let $S_n = a_1 + a_2 + \cdots + a_n$, S_n is non-decreasing and bdd by L. it must converge. Now $\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} a_n$ converges. ✱.

Examples 1. $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$ $\qquad \sum \frac{1}{3^n}$

2. $\sum \frac{1}{2+\sqrt{n}}$ $\qquad \sum \frac{1}{n}$ $\quad n \ge 4$.

Thm 9.13 (Limit comparison test).

Suppose that $a_n > 0$, $b_n > 0$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \text{ is finite and positive.}$$

Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Pf. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, we have $0 < \frac{a_n}{b_n} < L+1$ for some $n \ge N$.

$$0 < a_n < (L+1) b_n$$

by direct comparison test, convergence of $\sum b_n$ implies convergence of $\sum a_n$. Since $\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{L}$, convergence of $\sum a_n$ implies convergence of $\sum b_n$.

Examples 1. $\sum \frac{1}{an+b}$ $\quad a>0$, $b>0$.

$\sum \frac{1}{3n^2-4n+5}$, $\sum \frac{1}{\sqrt{3n-2}}$, $\sum \frac{n^2-10}{4n^5+n^3}$

2. $\sum \frac{\sqrt{n}}{n^2+1}$, 3. $\sum \frac{n\,2^n}{4n^3+1} \leftrightarrow \sum \frac{2^n}{n^2}$

§9.5. Alternating Series.

An infinite series is <u>alternating</u> if its terms alternate in sign.

e.g. the geometric series

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = 1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$$

is an alternating geometric series with $r=-\frac{1}{2}$.

<u>Thm 9.14</u> Alternating series test.

Let $a_n>0$, the alternating series

$$\sum_{n=1}^{\infty}(-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty}(-1)^{n+1}a_n$$

converge if the following two conditions are met.

1. $\lim_{n\to\infty} a_n = 0$ 2. $a_{n+1}\le a_n$ for all n.

Pf. Consider the partial sum.

$$S_{2n} = (a_1-a_2)+\cdots+(a_{2n-1}-a_{2n})$$

has all nonnegative terms and is $\{S_n\}$ is an nondecreasing seq.

Now $$S_{2n} = a_1-(a_2-a_3)-\cdots-(a_{2n-2}-a_{2n-1})-a_{2n}.$$

This implies $S_{2n}\le a_1$. So $\{S_{2n}\}$ is a bdd, nondecreasing seq hence converges to some value L. Now

$$S_{2n-1} = S_{2n}-a_{2n}$$

and ~~b~~ $a_{2n}\to 0$. So $\lim S_{2n-1} = \lim S_{2n} - \lim a_{2n} = \lim S_{2n} = L$

<u>Examples 1.2.3.</u>

#.

<u>Thm 9.15</u> Alternating series ~~with~~ remainder.

If a conv. alternating series satisfying the condition $a_{n+1}\le a_n$, then the remain $R_N = S-S_N$ can be estimated by

$$|S-S_N| = |R_N| \le a_{N+1}.$$

Absolute and conditional convergence.

<u>Def</u> 1. $\sum a_n$ is absolute convergent if $\sum |a_n|$ converges

2. $\sum a_n$ is conditional convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

<u>Thm 9.16</u> If $\sum |a_n|$ converges, then $\sum a_n$ converges.

Pf.
$$0 \le a_n + |a_n| \le 2|a_n|$$
by comparison test. $\sum (a_n + |a_n|)$ converges.

Now $a_n = (a_n + |a_n|) - |a_n|$, so
$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n| \quad \text{is convergent.}$$

<u>Example 5.6.</u>

§9.6. The ratio and root test.

<u>Thm 9.17</u> (ratio test)

$\sum a_n$ series with nonzero terms

1. $\sum a_n$ converges absolutely if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$
2. $\sum a_n$ diverges if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$
3. ~~$\sum a_n$~~ The ratio test is inconclusive if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

Pf. Assume that $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r < 1$. Choose R s.t. $0 \le r < R < 1$, then by definition of limit, we can choose N s.t. for all $n \ge N$, $\left|\frac{a_{n+1}}{a_n}\right| < R$. This means we have for $n \ge N$
$$|a_{N+1}| < R\,|a_N|$$
$$|a_{N+2}| < R\,|a_{N+1}| < R^2 |a_N|$$
So
$$\sum_{n=1}^{\infty} |a_{N+n}| \le |a_N| \sum_{n=1}^{\infty} R^n \quad \text{is convergent and hence}$$
$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{n=1}^{\infty} |a_{N+n}| \quad \text{is convergent.}$$
∎

<u>Examples</u> 1. 2. 3

<u>Thm 9.18</u> (root test)

$\sum a_n$ be a series

1. $\sum a_n$ converges absolutely if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1$
2. $\sum a_n$ diverges if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$
3. The root test is inconclusive if $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$

<u>Examples</u> 4, 5

§9.7 Taylor polynomials and approximations.

- Approximation of a function to 1^{st} order.
- Consider a polynomial of degree n $P(x)$ which is a degree ~~p~~ n approximation of a function $f(x)$. at a point $x=c$.

Write $$P_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n.$$

$$P_n(c) = a_0 = f(c).$$
$$P_n'(c) = a_1 = f'(c)$$
$$P_n''(c) = 2a_2 = f''(c)$$
$$\vdots$$
$$P_n^{(n)}(c) = 1\cdot 2\cdot 3\cdots n\cdot a_n = f^{n}(c).$$

This replies $P_n(x)$ can be written as
$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \qquad * \ 0! = 1.$$

This motivate the following

Def If f has n derivatives at c, the polynomial
$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$
is called the <u>Taylor polynomial</u> for f at c.

For $c=0$, the polynomial
$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$
is called the <u>Maclaurin polynomial</u> for f.

<u>Examples</u> 3, 4, 5. 6. 7.

<u>Remainder</u>. The difference $R_n(x)$ $f(x) - P_n(x)$ is called the remainder of the approximation where $P_n(x)$ is the Taylor polynomial for f.

We have

<u>Thm 9.19</u> (Taylor's thm)

If $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$ is the Taylor polynomial for f at c.

$R_n(x) = f(x) - P_n(x)$ is the remainder, then
$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$
where z is a number between x and c.

<u>Examples</u> 8, 9

§9.8 Power Series

Def If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

is called a power series. More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

is called a power series centered at c, where c is a const.

— Radius and interval of convergence.

A power series $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ can be seen as a fcn. of x.
For $f(x)$ to be well-defined, we have to determine the points (interval) where ~~f(x)~~ $\sum_{n=0}^{\infty} a_n (x-c)^n$ is convergent.

Thm 9.20 For a power series centered at c, precisely one of the following is true

1. The series converges only at c.
2. There is $R > 0$ s.t. the series ~~the series~~ converges absolutely for $|x-c| < R$ and diverges for $|x-c| > R$.
3. The series converges for all x.

R is called the radius of convergence

$1 \Leftrightarrow R = 0$
2
$3. \Leftrightarrow R = \infty.$

The set of value for which the power series converges is the interval of convergence

Examples 2, 3, 4

At the endpoints of the interval of convergence, we have to handle it by evaluating the endpoints.

Examples 5. 6. 7.

Differentiation and integration of power series.

Thm 9.21 If the func. given by

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

has a radius of conv. $R>0$ on the interval $(c-R, c+R)$. Then f is differentiable on $(c-R, c+R)$. Moreover, we have

1. $f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1}$

2. $\int f(x)dx = C + \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1}$

The radius of convergence of the above two series ~~are~~ is the same as the original ~~series~~, but at the endpoints the situation may be different.

Example 8

§9.9 Representation of functions by power series.

Geometric power series

$f(x) = \frac{1}{1-x}$

Consider the geom. series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$, $|r|<1$, If we set $a=1, r=x$

we have the power series representation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\cdots \qquad |x|<1$$

Note that the series represents $\frac{1}{1-x}$ only on the interval $(-1,1)$. To represent f in another interval, we have to find another series.

To get the series centered at -1,

$$\frac{1}{1-x} = \frac{1}{2-(x+1)} = \frac{\frac{1}{2}}{1-\frac{x+1}{2}} = \frac{a}{1-r} \quad , a=\frac{1}{2}, r=\frac{x+1}{2}, \text{ So for } |x+1|<2$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{x+1}{2}\right)^n = \frac{1}{2}\left[1+\frac{x+1}{2}+\frac{(x+1)^2}{4}+\frac{(x+1)^3}{8}+\cdots\right], \quad |x+1|<2.$$

Examples 1, 2

Let $f(x) = \sum a_n x^n$. $g(x) = \sum b_n x^n$

1. $f(kx) = \sum_{n=0}^{\infty} a_n k^n x^n$
2. $f(x^N) = \sum_{n=0}^{\infty} a_n x^{nN}$
3. $f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

The operation can change the interval of convergence, for example

$$\sum x^n + \sum \left(\frac{x}{2}\right)^n = \sum \left(1 + \frac{1}{2^n}\right) x^n$$
$$(-1,1) \cap (-2,2) \qquad (-1,1)$$

Example 3, 4, 5, 6

§9.10 Taylor and Maclaurin series.

Thm 9.22 If f is represented by a power series $f(x) = \sum a_n (x-c)^n$ for all x in an open interval I containing c, then $a_n = \frac{f^{(n)}(c)}{n!}$

and $f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$

Pf. $f^{(0)}(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + \cdots$

$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$

$f^{(n)}(x) = n!\, a_n + (n+1)!\, a_{n+1}(x-c) + \cdots$

$f^{(n)}(c) = n!\, a_n$

Def. The series $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ Taylor series for $f(x)$ at c,

If $c=0$, the series is the Maclaurin series for f.

Example 1

Thm 9.23 Conv. of Taylor series

If $\lim_{n\to\infty} R_n(x) = 0$. $\left(R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} \quad z \text{ is between } x \text{ and } c \right)$ then the Taylor series of f converges for all x in an interval I. then the Taylor series of f converges for all x in an interval I,

and equals $f(x)$, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$

Pf. Let $S_n(x)$ be the partial sum seq. of the Taylor poly. Then

$$S_n(x) = f(x) - R_n(x)$$

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \left(f(x) - R_n(x)\right) = f(x) - \lim_{n\to\infty} R_n(x)$$

So the Taylor series (the seq. of partial sum) converges to $f(x)$ iff $\lim_{n\to\infty} R_n(x) = 0$. #

<u>Example 2</u>, 3, 4, 5, 6, 8.