Ch 3 Applications of differentiation

§3.1 Extrema on an interval.

Def (extrema).
f: function defined on an interval I containing c.

1. $f(c)$ is the minimum of f on I if $f(c) \le f(x)$, $\forall x \in I$.
2. $f(c)$ is the maximum of f on I if $f(c) \ge f(x)$, $\forall x \in I$

They are also called the extreme values, absolute min or absolute max.

Thm 3.1 (extreme value thm) If f is cts on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

Def (relative extrema)

1. If $\exists$ open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a relative maximum of f, or f has a relative maximum at $(c, f(c))$.
2. If $\exists$ open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a relative minimum of f, or f has a relative minimum at $(c, f(c))$.

Def (critical numbers)
Let f be defined on c. If $f'(c) = 0$ or f is NOT differentiable at c, then c is a critical number of f.

Thm 3.2 If f has a relative min or relative maximum at c, then c is a critical number of f.

Pf. (i) If f is NOT differentiable, then by definition c is a critical number.

(ii) If f is differentiable at $x = c$, $f'(c)$ is >0, $=0$ or <0.
If $f'(c) > 0$, i.e. $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0$. This implies that $\frac{f(x) - f(c)}{x - c} > 0$ in a nbd of c, $x \ne c$.
left of c: $x < c$, & $f(x) < f(c) \Rightarrow f(c)$ is NOT a relative minimum.
right of c: $x > c$ & $f(x) > f(c) \Rightarrow f(c)$ is NOT a relative maximum.
So, $f'(c) > 0$ contradicts the hypothesis that $f(c)$ is a relative extremum.

Finding extrema on a closed interval $[a, b]$

1. Find the critical numbers of f in (a, b).
2. Evaluate ~~each~~ f at each critical number
3. Evaluate f at the endpoints a and b.
4. The least of these is the minimum and The largest is the maximum.

__Example__ Find the extrema of $f(x) = 2x - 3x^{2/3}$ on $[-1, 3]$. p168.

§3.2 Rolle's thm and the MVT.

__Thm 3.3__ (Rolle) f: cts on $[a, b]$ and differentiable on (a, b).
If $f(a) = f(b)$, then $\exists\ c \in (a, b)$ s.t. $f'(c) = 0$.

Pf a) If $f(x) = d = f(a) = f(b)$, f is const. on the interval $[a, b]$, $f'(x) = 0$ for any $x \in (a, b)$.

b) ~~By~~ If $f(x) > d$ for some $x \in (a, b)$. By extreme value thm, f has a maximum at some pt. c in the interval. Since $f(c) > d$, the maximum is NOT at the endpoints. This implies that $f(c)$ is relative maximum, by Thm $f'(c) = 0$, since f is differentiable at c.

c) If $f(x) < d$ for some $x \in (a, b)$, we get a relative minimum by the argument above. ※.

__Thm 3.4__ (MVT)
If f is cts on $[a, b]$ and f is differentiable on (a, b), then $\exists\ c \in (a, b)$ s.t.
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Pf Define the function $g(x)$ by
$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a),$$
Then $g(a) = 0 = g(b)$, g is cts on $[a, b]$ and is diff. on (a, b).
By Rolle's thm $\exists\ c \in (a, b)$ s.t. $g'(c) = 0$. Now
$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$
i.e.
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Ex 3. p195.

§3.3 Increasing and decreasing functions and the 1st derivative test.

Def A function f is

increasing on an interval I if for any $x_1, x_2 \in I$,
$x_1 < x_2$ implies that $f(x_1) < f(x_2)$.

decreasing on an interval I if for any $x_1, x_2 \in I$,
$x_1 < x_2$ implies that $f(x_1) > f(x_2)$.

Thm 3.5 Test for increasing and decreasing.

f: cts on $[a, b]$, differentiable on (a, b).

1. If $f'(x) > 0 \quad \forall\, x \in (a, b)$, then f is increasing on $[a, b]$.
2. If $f'(x) < 0 \quad \forall\, x \in (a, b)$, then f is decreasing on $[a, b]$.
3. If $f'(x) = 0 \quad \forall\, x \in (a, b)$, then f is constant on $[a, b]$.

Pf. If $f'(x) > 0$, $\forall\, x \in (a, b)$. For any $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. By MVT $\exists\; x_1 < c < x_2$, s.t.

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0.$$

$\Rightarrow f(x_2) > f(x_1)$, i.e. $f(x)$ is increasing on (a, b). ※.

Example 1 p180.

Finding functions on which a function is increasing or decreasing.

1. Locate the critical numbers of f in (a, b), and use number to determine the test intervals.
2. In each of the intervals, determine the sign of $f'(x)$ at one test number.
3. Use the above thm 3.5.

Thm 3.6 c is a critical number of f that is cts on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then

1. If $f'(x)$ changes from negative to positive at c, then f has a relative min at $(c, f(c))$.
2. If $f'(x)$ changes from positive to negative at c, then f has a relative max at $(c, f'(c))$.
3. If $f'(x)$ is positive on both sides of c or is negative on both sides of c, then $f(c)$ is neither a relative min nor a relative max.

Examples 3 & 4 p183 ~

§3.4 Concavity and 2^{nd} derivative test.

Def f: differentiable on I,
The graph of f is concave upward on I if f' is increasing on I
concave downward f' is decreasing on I.

The above are equivalent to the following:

f is concave upward on I $\Leftrightarrow$ the graph of f lies above of its tangent lines on I.

concave downward on I $\Rightarrow$ -- ——— below of its tangent lines on I.

Thm 3.7 (Test for concavity)
f function on I whose 2^{nd} derivative exists on I.

1 If $f''(x)>0$ for all $x \in I$, then f is concave upward on I

2 If $f''(x)<0$ for all $x \in I$, then f is concave downward on I.

Def f: cts on open interval I $c \in I$.
If the graph of f has a tangent line at the point $(c, f(c))$, then the point $(c, f(c))$ is a point of inflection of f is the concavity of f changes from upward to downward (or downward to upward).

Thm 3.8 If $(c, f(c))$ is a point of inflection, then either $f''(c)=0$ or f'' does NOT exist at $x=c$.

Example $f(x) = x^4 - 4x^3$. Find the points of inflection

$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x = 12x(x-2)$.

$f''(x)=0 \Rightarrow x = 0, 2$

	$-\infty < x < 0$	$0<x<2$	$2<x<\infty$
test value	$x=-1$	$x=1$	$x=3$
	$f''(-1) > 0$	$f''(1)<0$	$f''(3)>0$
	upward	downward	upward.

Thm 3.9 2^{nd} derivative test

f : a function s.t. $f'(c)=0$ and $f''(x)$ exists on an open interval containing c.

1 If $f''(c)>0$, then f has a relative minimum at $(c, f(c))$

2 If $f''(c)<0$, then f - ——— maximum at $(c, f(c))$.

If $f''(c)=0$, the test fails. That is, f ~~is~~ may have a relative maximum, relative minimum or neither. In this case, we should use 1^{st} derivative test.

§3.5 Limits at infinity.

Def L : a real number.

1. $\lim_{x\to\infty} f(x) = L$ means that for every $\varepsilon>0$, $\exists$ $M>0$ s.t. $|f(x)-L|<\varepsilon$ whenever $x>M$

2. $\lim_{x\to-\infty} f(x) = L$ $\varepsilon>0$ $\exists$ $M<0$ s.t. $|f(x)-L|<\varepsilon$ whenever $x<N$.

In both cases, we say that $y=L$ is a <u>horizontal asymptote</u> for the function $f(x)$ as $x\to\infty$.

Thm 3.10 Limits at infinity

If $r>0$ rational number, c is any real number, then

$$\lim_{x\to\infty}\frac{c}{x^r}=0$$

If x^r is defined for $x<0$, then

$$\lim_{x\to-\infty}\frac{c}{x^r}=0.$$

If $\lim_{x\to\infty} f(x)$, $\lim_{x\to\infty} g(x)$ exist, then Ex 3.4.5.

$$\lim_{x\to\infty}(f(x)+g(x)) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$$

$$\lim_{x\to\infty}(f(x)\, g(x)) = \lim_{x\to\infty} f(x)\cdot \lim_{x\to\infty} g(x)$$

Def 1. $\lim_{x\to\infty} f(x) = \infty$ means. $f(x)>M$ whenever $x>N$.

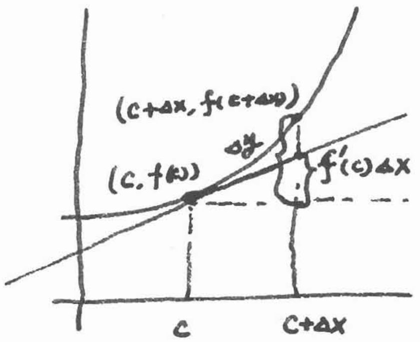
§3.6 A summary of curve sketching

Guidelines:

1. Determine the domain and range of the function
2. Determine the intercepts, asymptotes and symmetry of the graph.
3. Locate the x-values for which $f'(x)$ and $f''(x)$ either are zero or do NOT exist. Use these results to determine relative extrema and points of inflection.

<u>Examples</u> 1. 2. 4. 5.

§3.9 Differentials.



The tangent line at $(c, f(c))$ is

$$y = f(c) + f'(c)(x-c)$$

when $x - c = \Delta x$ is very small, the change $\Delta y = f(c+\Delta x) - f(c)$ can be approximated by

$$\Delta y = f(c+\Delta x) - f(c) \approx f'(c)\Delta x. \longrightarrow (1)$$

When $\Delta x \to 0$, denote it by dx, we then define the differential to

$$dy = f'(x)\,dx.$$

Ex 2, before 4, 5.

(1) $\Rightarrow$ $f(x+\Delta x) \approx f(x) + dy = f(x) + f'(x)\,dx.$

6 $\sqrt{16.5}$

Ch4 Integration

§4.1 Antiderivative and indefinite integral.

Def A function F is "an" antiderivative of f on the interval I if $F'(x) = f(x)$ for all $x \in I$.

Thm 4.1 If F is an antiderivative of f, then G is an antiderivative of f on the interval I iff $G(x) = F(x) + C$, where C is a const.

pf Clearly $F(x)+C$ is also an antiderivative of f.

$\Leftarrow$ If G is an antiderivative of f. Let

$$H(x) = G(x) - F(x).$$

If H is NOT const. $\exists$ $a<b$ in I s.t. $H(a) \neq H(b)$ and by MVT, $\exists$ $a<c<b$ s.t.

$$H'(c) = \frac{H(b)-H(a)}{b-a} \neq 0.$$

Now since G, F are anti-derivatives of f $G'(x) = F'(x) = f(x)$.

In particular $G'(c) = F'(c)$, hence $H'(c) = 0$.

This contradiction implies that $H(x) = \overset{C}{\text{const.}}$ and

$$G(x) = F(x) + C.$$

We use the notation

$$\int f(x)dx$$

for anti-derivative of f. It is also called <u>indefinite integral.</u>

Basic integration rules:

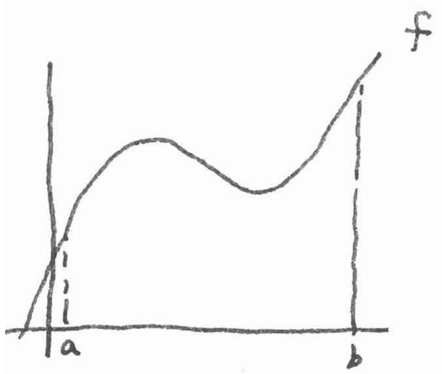
1. $\int F'(x)dx = F(x) + C.$
2. $\frac{d}{dx} \int f(x)dx = f(x)$

— p.250. table.

—— Examples 3, 4. 5 .6.

§4.2 Area.

Upper and lower sum.



Approximation of the area

1 Subdivide the interval $[a, b]$ into n subintervals with width $\Delta x = \frac{b-a}{n}$

2. In each subinterval

$f(m_i) =$ min. value of $f(x)$ in the i^{th} subinterval

$f(M_i) =$ max value of $f(x)$ in the i^{th} subinterval

(can be done by extreme value thm of for cts func. fcn on closed intervals).

The lower sum is defined to be

$$\text{lower sum} = s(n) = \sum_{i=1}^{n} f(m_i)\Delta x$$

$$\text{upper sum} = S(n) = \sum_{i=1}^{n} f(M_i)\Delta x.$$

$$s(n) \le \text{Area of the region} \le S(n)$$

<u>Thm 4.3</u> f cts nonnegative on $[a, b]$

Then $\lim_{n\to\infty} s(n) = \lim_{n\to\infty} S(n)$.

Note that for any x in the i^{th} interval $f(m_i) \le f(x) \le f(M_i)$, and since both limit in Thm 4.3 are equal, by squeeze thm, we can make the following

<u>Def</u> f: cts and nonnegative on the interval $[a, b]$. The area of the region bounded by the graph of f, the x-axis and the two lines $x=a$, $x=b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \le c_i \le x_i.$$

where $\Delta x = \frac{b-a}{n}$.

Ex 6 p266.

§4.3 Riemann sums and definite integrals.

Riemann sum

Def f : defined on a closed ~~integ~~ interval $[a.b]$.
Δ : partition of $[a.b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Δx_i : the width of the i^{th} ~~interval~~. subinterval.
c_i : any point in the i^{th} sub-interval

Then the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x_i \qquad x_{i-1} \le c_i \le x_i$$

is called the Riemann sum of f for the partition Δ.

$\|\Delta\|$: width of the largest subinterval in the partition Δ.
Note that $\|\Delta\| \to 0$ as $n \to \infty$.

Def If f is defined on $[a.b]$ and the limit

$$\lim_{\|\Delta\| \to 0} \sum f(c_i)\Delta x_i$$

exists, the f is integrable on $[a.b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum f(c_i)\Delta x_i = \int_a^b f(x)dx$$

and it is called the definite integral of f from a to b.
a : lower limit
b : upper limit.

Thm 4.4 If f is cts on $[a.b]$, then f is integrable on $[a.b]$.

Thm 4.5 f : cts, nonnegative on $[a.b]$, then

$$\text{Area} = \int_a^b f(x)\,dx.$$

Properties of definite integrals.

Def 1. If f is defined at $x=a$, then we set $\int_a^a f(x)dx = 0$.

2. If f is integrable on $[a,b]$, then we set $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

Thm 4.6 If f is integrable on the three closed ~~interv~~ interval determined by a, b, c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Thm 4.7 If f, g are integrable on $[a,b]$, k any const. then

1. $\int_a^b kf(x)\,dx = k\int_a^b f(x)dx$

2. $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$

Thm 4.8 1. If f is integrable and nonnegative on $[a,b]$, then

$$0 \le \int_a^b f(x)dx$$

2. If f and g are integrable and $f(x) \ge g(x)$ for all x in $[a,b]$,

$$\int_a^b f(x)dx \ge \int_a^b g(x)dx.$$