Ch 2 Differentiation

§2.1 The derivative and the tangent line problem.

<u>Def.</u> If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the <u>tangent line</u> to the graph of f at $(c, f(c))$.

The <u>derivative</u> of f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which the limit exists, f' is a function of x.

<u>Remark</u>: The process of finding the derivative of a fun. is called <u>differentiation</u>. A fun. f is <u>differentiable</u> at x if $f'(x)$ exists. f is called differentiable on (a, b), if it is differentiable at each pt. x in (a, b).

<u>Notation</u>: $f'(x)$, $\frac{dy}{dx}$, y', $\frac{d}{dx} f$

<u>Examples</u> a) $f(x) = x^2+1$, b) $f(x) = \sqrt{x}$.

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$$

<u>Differentiability</u>:

Not differentiable if ① $\lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} \neq \lim_{x \to c^+} \frac{f(x)-f(c)}{x-c}$

or ② $\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = \pm\infty$.

<u>Examples</u> a) $f(x) = |x-2|$ b) $f(x) = x^{1/3}$

<u>Thm 2.1</u> If f is differentiable at $x=c$, f is cts at $x=c$

§2.2 Basic differentiation rule.

Thm 2.2 If c is a real number, then

$$\frac{d}{dx} c = 0.$$

Thm 2.3 (Power rule)

If n is any rational number, then $f(x) = x^n$ is differentiable and

$$f'(x) = n x^{n-1}$$

In particular, we have $\frac{d}{dx} x = 1$.

Thm 2.4 If f is differentiable and c is any real number, then

$$\frac{d}{dx} (cf(x)) = c f'(x).$$

Thm 2.5 If $f(x)$ and $g(x)$ are differentiable, then $f \pm g$ are differentiable and

$$\frac{d}{dx} (f \pm g)(x) = f'(x) \pm g'(x).$$

Examples a) $f(x) = \frac{1}{x^2}$ b) $f(x) = x^3 - 4x + 5$.

Thm 2.6 $\frac{d}{dx} \sin x = \cos x$. $\frac{d}{dx} \cos x = -\sin x$.

§2.3 Product and quotient rules

Thm 2.7 (Product rule)

If $f(x)$ and $g(x)$ are differentiable, then

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

pf.

Thm 2.8 (quotient rule)

If $f(x)$, $g(x)$ are differentiable with $g(x) \neq 0$.

then $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

pf.

Examples Find the derivatives

1. $h(x) = (3x-2x^2)(5+4x)$.
2. $y = \dfrac{5x-2}{x^2+1}$.
3. $y = 3x^2 \sin x$
4. The tangent line of $f(x) = \dfrac{3-\frac{1}{x}}{x+5}$ at $(-1, 1)$.
5. $y = \dfrac{1-\cos x}{\sin x} = \csc x - \cot x$.

Higher order derivatives.

1^{st} order	y',	$f'(x)$	$\frac{dy}{dx}$,	$\frac{df}{dx}$
2^{nd} order	y'',	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}f$
n^{th} order	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	...

§2.4 Chain Rule.

Thm 2.10 If $y=f(u)$ is a differentiable function of u and $u=g(x)$ is a differentiable function of x. then $y=f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

or $\quad \frac{d}{dx}f(g(x))=f'(g(x))\cdot g'(x).$

Examples Find $\frac{dy}{dx}$ if

(1) $y=(x^2+1)^3$

(2) general power rule $y=(u(x))^n$. n rational, $u(x)$ is differentiable

then $\frac{dy}{dx}=n\,u(x)^{n-1}\frac{du}{dx}$.

(3) $f(x)=\sqrt[3]{(x^2-1)^2}$

(4) $g(t)=\frac{-7}{(2t-3)^2}$ (5) $f(x)=\frac{x}{\sqrt[3]{x^2+4}}$

(6) $y=\left(\frac{3x-1}{x^2+3}\right)^2$

(7) $\frac{d}{dx}\sin u=\cos u\frac{du}{dx}$ $\quad \frac{d}{dx}\cos u=-\sin u\frac{du}{dx}$

$\frac{d}{dx}\tan u=\sec^2 u\frac{du}{dx}$ $\quad \frac{d}{dx}\cot u=-\csc^2 u\frac{du}{dx}$

(8) $y=\cos^3 x$

(9) $f(t)=\sin^3 4t$.

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§2.5 Implicit Differentiation.

If the function y of x is given implicitly by

$$f(x,y)=0,$$

we can still find $\frac{dy}{dx}$ in terms of $f(x,y)$ and the chain rule.

Examples 2. 4. 5. 6. 7.