Ch 1 Limits.

§1.2 Finding limits graphically

Examples 1. $f(x) = \dfrac{x^3-1}{x-1}$, $x \neq 1$.

The behavior of f as ~~f~~ x approaches 1.

$$\lim_{x \to 1} f(x) = 3.$$

2. $f(x) = \dfrac{x}{\sqrt{x+1}-1}$ $\qquad \lim_{x \to 0} f(x) = 2.$

The behavior of $f(x)$ near $x=0$.

3. Let $f(x) = \begin{cases} 1 & x \neq 2 \\ 0 & x = 2. \end{cases}$ $\qquad \lim_{x \to 2} f(x) = 1.$ but $f(2) = 0.$

Examples (Limits that fail to exist)

1. $\lim_{x \to 0} \dfrac{|x|}{x}$

2. $\lim_{x \to 0} \dfrac{1}{x^2}$

3. $\lim_{x \to 0} \sin \dfrac{1}{x}$

Formal definition of limit.

$\lim_{x \to c} f(x) = L$ $\qquad$ f becomes arbitrarily close to L as x approaches c.

$\varepsilon - \delta$ definition $\qquad |f(x) - L| < \varepsilon$, $0 < |x-c| < \delta$.

for each ε, $\exists$ δ s.t.

§1.3 Finding limits analytically.

Thm 1.1 b, c real numbers, n: positive integer.

(i) $\lim_{x\to c} b = b$. (ii) $\lim_{x\to c} x = c$. (iii) $\lim_{x\to c} x^n = c^n$.

Thm 1.2 f, g functions with limits $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = K$.

1. $\lim_{x\to c} b f(x) = bL$
2. $\lim_{x\to c} f(x) \pm g(x) = L \pm K$.
3. $\lim_{x\to c} f(x) g(x) = LK$
4. $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K}$ if $K \neq 0$.
5. $\lim_{x\to c} (f(x))^n = L^n$

Thm 1.3

$\Rightarrow$ (i) If $p(x)$ is a polynomial, then $\lim_{x\to c} p(x) = p(c)$.

(ii) If $r(x) = \frac{p(x)}{g(x)}$, where $p(x), g(x)$ are polynomials.

c is a number s.t. $g(c) \neq 0$, then

$$\lim_{x\to c} r(x) = \lim_{x\to c} \frac{p(x)}{g(x)} = \frac{p(c)}{g(c)}.$$

Thm 1.4 $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$. n is odd true for all c; n is even for $c > 0$.

Thm 1.5 If $\lim_{x\to c} g(x) = L$, $\lim_{x\to L} f(x) = f(L)$. then

$$\lim_{x\to c} f(g(x)) = f(\lim g(x)) = f(L)$$

Thm 1.6 (i) $\lim_{x\to c} \sin x = \sin c$, (ii) $\lim_{x\to c} \cos x = \cos c$

(iii) $\lim_{x\to c} \tan x = \tan x$ (iv) $\lim_{x\to c} \cot x = \cot c$.

(v) $\lim_{x\to c} \sec x = \sec c$ (vi) $\lim_{x\to c} \csc x = \csc c$.

Thm 1.7 c real number. and $f(x) = g(x)$ $\forall x \neq c$. in an open interval of c. If $\lim_{x\to c} g(x)$ exists, then

$\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$

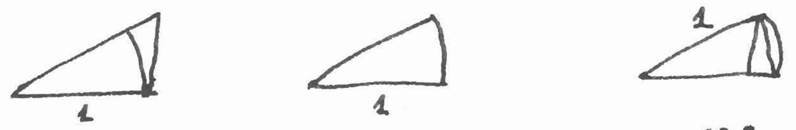
Examples $\lim_{x\to 1} \frac{x^3-1}{x-1}$. $\lim_{x\to -3} \frac{x^2+x-6}{x+3}$.

$\lim \frac{\sqrt{x+1} - 1}{x}$

<u>Thm 1.8</u> (The squeeze theorem)

If $h(x) \le f(x) \le g(x)$ in an open interval containing c, except possibly at c and if $\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$ then

<u>Example</u> 1 $\lim_{x\to 0} \frac{\sin x}{x} = 1$ $\qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$



A $\quad \frac{\tan\theta}{2} \quad \ge \quad \frac{\theta}{2} \quad \ge \quad \frac{\sin\theta}{2}$

$$\frac{1}{\cos\theta} \ge \frac{\theta}{\sin\theta} \ge 1.$$

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1.$$

Now apply squeeze thm.

2. $\lim \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \frac{1}{\cos x} = \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{1}{\cos x} = 1.$

3. $\lim_{x\to 0} \frac{\sin 4x}{x} = 4 \lim_{x\to 0} \frac{\sin 4x}{4x} = 4 \lim_{y\to 0} \frac{\sin y}{y} = 4.$

§1.4. Continuity and one-side limits.

Def A function f is cts at c if
$f(c)$ is defined. $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} f(x) = f(c)$.

The function f is continuous on an open interval (a, b) if it is continuous at each point in (a,b).

Discontinuity: jump, infinity, not defined.

<u>One-side limits</u>

limit from the right $\lim_{x\to c^+} f(x) = L$

left $\lim_{x\to c^-} f(x) = L$.

<u>Example</u> greatest integer function $f(x) = [x]$

$\lim_{x\to 0^-} [x] = -1$. $\lim_{x\to 0^+} [x] = 0$.

<u>Thm 1.10</u> f: function, c, L real numbers.

Then $\lim_{x\to c} f(x) = L$ iff $\lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$.

<u>Def</u> A function f is cts on the closed interval $[a, b]$ if f is continuous on (a, b) and

$\lim_{x\to a^+} f(x) = f(a)$, $\lim_{x\to b^-} f(x) = f(b)$.

$f(x) = \sqrt{1-x^2}$

<u>Example</u> $\lim_{x\to -1^+} \sqrt{1-x^2} = 0 = f(-1)$

$\lim_{x\to 1^-} \sqrt{1-x^2} = 0 = f(1)$

$f(x)$ is cts on $[-1, 1]$.

<u>Thm 1.11</u> b real number f, g are cts at $x=c$, then the following functions are cts

(a) bf.

(b) $f \pm g$.

(c) fg

(d) f/g if $g(c) \neq 0$.

<u>Thm 1.12</u> If g is cts at $x=c$, f is cts at $g(c)$, then the composite $(f\circ g)(x) = f(g(x))$ is cts at $x=c$.

Example. Test for continuity.

a) $\tan x = f(x)$. b) $f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$ c) $f(x) = x \sin\frac{1}{x}$

$-|x| \le x\sin\frac{1}{x} \le |x|$

Thm 1.13 (Intermediate value theorem)

If f is cts on the closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ s.t.

$$f(c) = k.$$

Example $f(x) = x^3 + 2x - 1$. $f(x)$ has a root in $[0,1]$.

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§1.5 Infinite Limits

Def: f: function defined in an open interval containing c (except possibly at c).

$\lim_{x\to c} f(x) = \infty$ means that for each $M > 0$, $\exists\ \delta > 0$ s.t. $f(x) > M$ whenever $0 < |x - c| < \delta$.

$\lim f(x) = -\infty$ means that for each $N < 0$. $\exists\ \delta > 0$ s.t. $f(x) < N$ whenever $0 < |x - c| < \delta$.

$\lim_{x\to c^-} f(x) = \infty$ means that $f(x) > M$ for $c - \delta < x < c$.

In all of the above cases, we say $x = c$ is a vertical asymptote.

Thm 1.14 If $h(x) = \dfrac{f(x)}{g(x)}$ where $f(x), g(x)$ are cts at $x = c$, with $f(c) \neq 0$ and $g(c) = 0$, then $h(x)$ has a vertical asymptote at $x = c$.

Examples a) $f(x) = \dfrac{1}{2(x+1)}$ b) $f(x) = \dfrac{x^2+1}{x^2-1}$ c) $f(x) = \cot x$.

d) $f(x) = \dfrac{x^2 + 2x - 8}{x^2 - 4}$.

Thm 1.15 If $\lim_{x\to c} f(x) = \infty$, $\lim_{x\to c} g(x) = L$

1. $\lim_{x\to c} (f(x) \pm g(x)) = \infty$
2. $\lim_{x\to c} f(x)g(x) = \begin{cases} \infty & \text{if } L > 0 \\ -\infty & \text{if } L < 0 \end{cases}$
3. $\lim_{x\to c} \dfrac{g(x)}{f(x)} = 0$.